THE PREPARATION OF PROJECTION DIAGRAMS

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In crystallographical, optical, and X-ray work with crystals, projection diagrams of several different types are widely used, especially for the graphical analysis of data of measurement. In crystallography, the gnomonic and stereographic projection plots serve the purpose best; in crystal optics, the stereographic, the

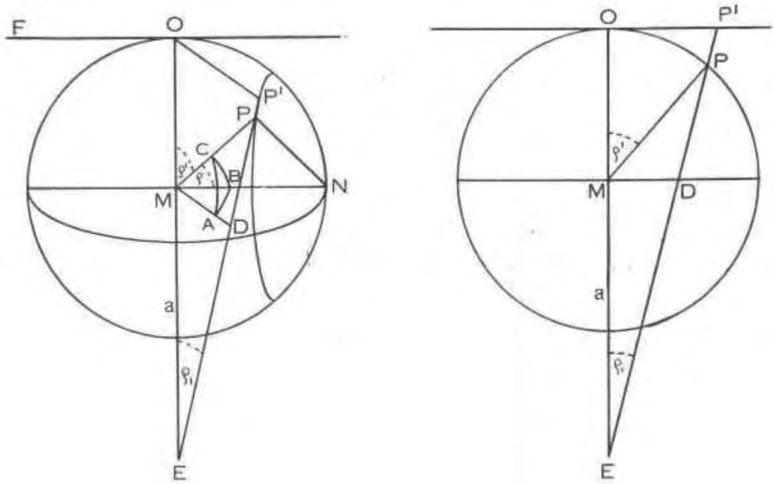

Fig. 1

Diagram used in the derivation of the formula for parallels of latitude in perspective projection diagrams. The point E is the eye of the observer; P, the point on the sphere projected to P' on the plane FOP', tangent to the sphere at O, or to D on the equatorial plane NMD normal to EM. The angle $NMP=\rho$, $OMP=\rho'$, $NMD=\phi$.

Fig. 1a

The plane EOP' of Fig. 1 drawn in the plane of the paper to show the angular relations.

orthographic, and the angle projection plots; in X-ray analysis, the reflection and the gnomonic projections. The preparation of these projection plots, which represent the sphere with its parallels of latitude and meridians of longitude, spaced either 1° or 2° apart is a tedious task; but once drawn, they can be reproduced and made available by publication. This has been done for the more important projections and the diagrams or nets have long been in use. It is of interest, however, to derive and to list the equations that form the basis for these projection plots and to compare the several projections from this viewpoint.

THE PERSPECTIVE PROJECTIONS. In these projections the eye of the observer is located at a definite point in space from which the points on the surface of the sphere are viewed. The points of intersection of the lines of sight with the fixed plane of projection are then the desired projection points.

SMALL CIRCLES. In Fig. 1 let M be the center of the sphere, MP a given radial direction which pierces the sphere at P, E the position of the eye, EP the line of sight to P', the projection point of P on the tangent plane normal to EO. If the point P lies on a small circle, such that the angle $PMN=\rho$, a definite given angle, an expression for the condition to be met can be deduced from the right angle spherical triangle ABC, in which CAB is a right angle, $BA=\phi$, $CB=\rho$ and $CA=90°-OMP=90-\rho^1$,

$$\cos\rho=\cos\phi\cdot\sin\rho^1. \qquad (1)$$

In the triangle MEP, (Fig. 1a) $MP=1$, $ME=a$ and angle $MPE=PMO-PEM=\rho^1-\rho_1$. Therefore

$$\frac{ME}{MP}=\frac{\sin MPE}{\sin MEP} \quad \text{or}$$

$$a=\frac{\sin(\rho^1-\rho_1)}{\sin\rho_1}=\sin\rho^1\cdot\cot\rho_1-\cos\rho^1. \qquad (2)$$

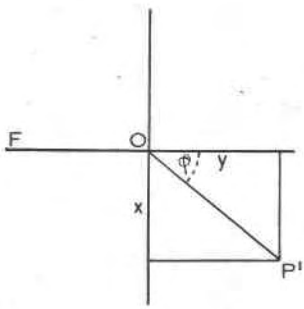

Fig. 1b
Diagram to show the relations in the projection plane FOP' of Fig. 1.

In the tangent plane of projection, the line $OP'=(1+a)\cdot\tan\rho_1$ and (Fig 1b)

$$x=(1+a) \tan \rho_1 . \sin \phi$$
$$y=(1+a) \tan \rho_1 . \cos \phi$$

Therefore

$$\cos \phi=\frac{y}{\sqrt{x^2+y^2}} \tag{3}$$

$$\tan \rho_1=\frac{\sqrt{x^2+y^2}}{1+a} . \tag{4}$$

From equations (1) and (2) we find

$$\cos \rho \cdot \cot \rho_1=a \cos \phi+\sqrt{\cos^2 \phi-\cos^2 \rho} \tag{5}$$

Combining this equation with (3) and (4) we obtain

$$(1+a) \cos \rho-a y=\sqrt{y^2-\cos^2 \rho \cdot (x^2+y^2)} \tag{6}$$

By assigning to a different values we ascertain for each perspective projection its appropriate equation.

In the *gnomonic* projection $a=0$, and (6) reduces to

$$y^2=\cot^2 \rho \cdot (1+x^2) \tag{6a}$$

the equation of an hyperbola.

In the *orthographic* projection $a=\infty$ and (6) becomes

$$y=\cos \rho \tag{6b}$$

the equation of a straight line.

In the *stereographic* projection $a=1$; the projection plane, is, moreover, the equatorial plane and the coordinates x', y' are half the normal values. Equation (6) reduces to

$$\cos \rho \cdot (4+x^2+y^2)=4y, \quad \text{or} \quad \cos \rho \cdot (1+x'^2+y'^2)=2y' \tag{6c}$$

or

$$\left(y'-\frac{1}{\cos^2 \rho}\right)^2+x'^2=\tan^2 \rho$$

the equation of a circle.

Great Circles. In Fig. 2 let $LPKN$ be a great circle, the plane of which includes the angle $KMO=\rho$ with the pole MO. Let P be a point on the great circle that is to be plotted in projection. In

the spherical triangle POK, $OK=\rho$, $OP=\rho^1$, $KOP=\phi$, and $PKO=90°$.
Therefore

$$\tan \rho = \cos \phi \tan \rho^1 \tag{7}$$

In triangle PMO equation (2) applies as heretofore

$$a = \frac{\sin(\rho^1 - \rho_1)}{\sin \rho_1} = \sin \rho^1 . \cot \rho_1 - \cos \rho^1 \tag{2}$$

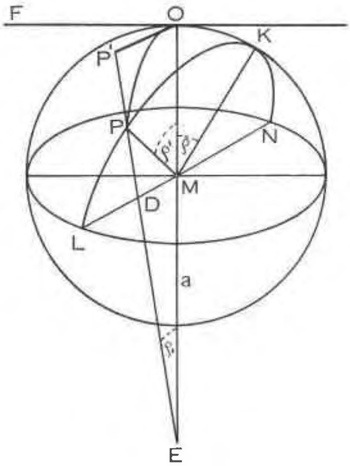

Fig. 2

Diagram used in the derivation of the formula for great circles in perspective projection. The point E is the eye of the observer; P' and D are the projection points of P respectively on the tangent and equatorial planes, normal to EM.

Combining (2) and (7) to eliminate ρ^1 we find,

$$a\sqrt{\tan^2 \rho + \cos^2 \phi} = \tan \rho . \cot \rho_1 - \cos \phi$$

and this equation together with (3) and (4) yields

$$(a+1) \cdot \tan \rho - y = a\sqrt{y^2 + \tan^2 \rho \cdot (x^2+y^2)} . \tag{8}$$

By assigning proper values of a for the several projections we find for the different projections:

Gnomonic projection. $a=0$

$$y = \tan \rho \tag{8a}$$

the equation of a straight line.
Orothographic projection. $a=\infty$

$$\frac{y^2}{\sin^2 \rho}+x^2=1 \tag{8b}$$

the equation of an ellipse.

Stereographic projection. Projection plane is the equatorial plane and coordinates x', y' are half the normal values; $a=1$

$$(y+2\cot \rho)^2+x^2=\frac{4}{\sin^2 \rho}$$

or

$$(y'+\cot \rho)^2+x'^2=\frac{1}{\sin^2 \rho} \tag{8c}$$

the equation of a circle.

The Reflection Projection

This is the projection in which the points of the diffraction pattern in the X-ray photograph of a single crystal appear. This projection bears the same relation to the gnomonic projection that the gnomonic projection bears to the stereographic. In each case, the azimuth angles, φ, remain the same; as ρ is the angle between the pole and the direction represented by the point in the projection, the distance of the point in projection from the pole is $\tan \rho/2$ in the *stereographic* projection, $\tan\rho$ in the *gnomonic*, and $\tan 2\rho$ in the *reflection* projection. The relations are illustrated in Fig. 3.

The equations for the small and great circles of the reflection projection plot are ascertained by the methods noted above.

Small circles. Equation(1)

$$\cos \rho=\cos \phi \cdot \sin \rho^1 \tag{1}$$

is valid as heretofore. Also in the projection plane we have

$$x=\tan 2\rho^1 \cdot \sin \phi$$

$$y=\tan 2\rho^1 . \cos \phi$$

$$\cos\phi = \frac{y}{\sqrt{x^2 + y^2}} \tag{9}$$

$$\tan 2\rho^1 = \sqrt{x^2 + y^2} \tag{10}$$

By combining equations 1, 9, and 10, we obtain the expression

$$y^4 = 4\cos^2\rho \cdot (1 + x^2 + y^2)\left[y^2 - \cos^2\rho.(x^2 + y^2)\right] \tag{11}$$

which represents a curve of the fourth degree.

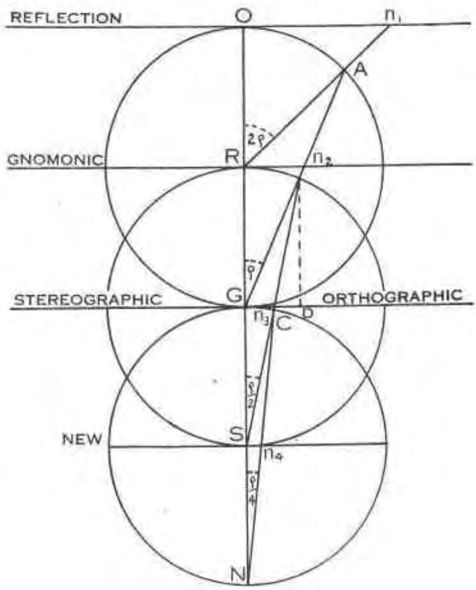

Fig. 3
Sectional diagram to illustrate the relations between the different projections listed.

Great circles. Equation (7)

$$\tan\rho = \cos\phi.\ \tan\rho^1 \tag{7}$$

combined with equations 9 and 10 yields the expression for the great circle curves in this projection

$$y^2(1 - \tan^2\rho) - \tan^2\rho \cdot x^2 = 2y\tan\rho \tag{12}$$

or

$$(2\cot 2\rho \cdot y - 1)^2 - (1 - \tan^2\rho)x^2 = 1 \tag{12a}$$

This is the equation of an hyperbola for angles $\rho < 45°$.

A new projection. If Fig. 3 were extended downward to include a third sphere a new projection would result that bears the same relation to the stereographic that the stereographic does to the gnomonic. It is not difficult to write down the equations for the great and small circles in this projection which projects the entire sphere within a circle of unit radius.

Small circles. For these circles on the sphere the relation expressed by equation 1 is valid

$$\cos\rho = \cos\phi \sin\rho^1. \tag{1}$$

Also the equations

$$x = \tan\frac{\rho}{4}\cdot\sin\phi$$

$$y = \tan\frac{\rho}{4}\cdot\cos\phi$$

$$\cos\phi = \frac{y}{\sqrt{x^2+y^2}} \tag{13}$$

$$\tan\frac{\rho^1}{4} = \sqrt{x^2+y^2} \tag{14}$$

From equations 1, 13, and 14 the expression can be deduced

$$(1+x^2+y^2)^2\cos\rho = 4y(1-x^2-y^2) \tag{15}$$

which represents a curve of the fourth degree.

Great circles. From equation 7

$$\tan\rho = \cos\phi.\ \tan\rho^1 \tag{7}$$

and from equations 13 and 14 we find the expression

$$(1-x^2-y^2)^2 - 4(x^2+y^2) = 4\cdot\cot\rho\cdot y(1-x^2-y^2) \tag{16}$$

which represents a curve of the fourth degree.

These equations are more complex than those of the foregoing projections. The projection is neither angle true nor area true and will be useful only rarely, where it is desired to project the major part of a sphere. It is, however, like the foregoing projections a type of perspective projection.

The new projection serves to project the entire sphere within a circle of unit radius. The stereographic projection represents the hemisphere in a circle of unit radius, and the whole sphere if the plane of projection is indefinitely extended. It is angle true; all zone lines are circular arcs. The gnomonic projection represents the hemisphere on a plane extending to infinity in all directions. On it all zone lines are straight lines. The reflection projection represents a spherical cone of 45°, if the plane of projection is indefinitely extended.