CALAVERITE AND THE LAW OF COMPLICATION

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Few crystallographers would venture to think that the Law of Simple Rational Intercepts and Indices is not a perfectly general truth; yet this is the conclusion reached by Victor Goldschmidt (Heidelberg), Charles Palache and the present writer[1] in a recently published account of an extended study of calaverite, the crystallized di-telluride of gold which has puzzled crystallographers since the end of the last century. Tersely written in German, assuming familiarity with the Goldschmidt method of crystal measurement and discussion, and making use of a new and significant crystallographic conception, this joint account may have presented some difficulties to the English-speaking reader. Encouraged by Professor Palache, the junior author has therefore prepared the present paper which gives the essential theory used in the joint paper and its application to the crystallography of calaverite, followed by a formal statement of the crystallography of this remarkable mineral. The opportunity is also taken to give a brief account and discussion of Professor Goldschmidt's application of the Law of Complication to fields outside of crystallography.

The Classical Laws of Crystallography

The goniometric study of crystals has led to three empirical generalizations which have long been accepted as the laws of systematic crystallography.

1. The Law of Constancy of Angles.
2. The Law of Zones.
3. The Law of Rationality.

The Law of Constancy of Angles states that the angular relations of the forms of crystals of any species are constant and proper to the species.

The Law of Zones states that crystal faces occur in groups in each of which the faces are parallel to a common axis.

By the Law of Rationality, the Law of Simple Rational Intercepts and Indices is generally understood. This law may be stated as follows: With suitably chosen axes and axial lengths, every face of a crystal is parallel to a plane which cuts one axis at the unit

[1] Ueber Calaverit, *N. J. f. Min.*, Beil. Bd. **63A**, pp. 1–58, figs. 1–22, pls. i–x, 1931.

length and cuts the other axes either at infinity or at distances from the axial centre which are simple rational multiples or fractions of the corresponding unit lengths; and therefore the angular relations of every face may be given, with reference to suitable crystal axes, by symbols involving, on any system of notation, only simple rational numbers, zero and infinity.

These laws determine the procedure of crystallographic investigation. A number of crystals of a species are measured, and it is observed that like forms on all the crystals have the same angular relations. The measurements are projected and the faces are found to lie in zones. Suitable axes are chosen and the appropriate simple rational symbols are given to all the observed plane faces. The consistent success of this procedure has consolidated the classical laws of crystallography and placed them in a position of apparently unassailable security.

The study of calaverite was naturally approached in this way; but repeated investigations[2] of excellent material showed that, while the first and second laws held strictly true, the third failed. It is not possible to choose axes for calaverite so that the majority of the forms receive simple symbols; and no admissible assumptions of twinned or heterogeneous structure will serve to bring this crystal species within the Law of Rationality in its generally accepted form. No one seriously thought that the Law of Rationality might be inadequate, and thus the tiny, brilliant, golden crystals of calaverite remained complete enigmas.

Goldschmidt's Law of Crystallography—The Law of Complication

In the gnomonic projection, which forms the graphical foundation of Professor Goldschmidt's system, the Law of Constancy of Angles is proved by the constancy of the co-ordinates of like nodes (projection points of face normals) in crystals of the same species. The Law of Zones is apparent in the fact that the nodes of any one crystal lie on a network of straight lines extending to infinity. And

[2] Of these only two were published: S. L. Penfield and W. E. Ford, *Amer. Jour. Sci.*, **xii**, pp. 225–246, 1901; G. F. H. Smith, *Min. Mag.*, **xiii**, pp. 122–150, 1902. Laborious unpublished studies by Palache, Goldschmidt, Moses, Goldschmidt and Görgey, Goldschmidt and Neff, and Palache and Peacock, representing efforts spread over more than thirty years, confirmed and extended the previous observations, but failed to reconcile them with the accepted laws of crystallography.

the Law of Rationality appears in the fact that the linear distances between the nodes in a zone are simple rational fractions or multiples of a suitably chosen unit length, when the plane of the projection is normal to a zone axis.

In any zone certain nodes, usually those corresponding to the largest and most frequent faces, may be recognized as principal nodes. In zones having one such node at infinity (∞), another being taken as the origin of measurement (0), and a third important node being taken to give the unit of spacing (1), it is found by statistical study[3] that the remaining nodes of the zone segment between the nodes (0) and (∞) occur most frequently at distances $^1/_2$, 2, less frequently at $^1/_3$, $^2/_3$, $^3/_2$, 3, and rarely at other distances from the end-node (0) (fig. 1). Thus, out of the great variety of simple rational spacings which might occur in a zone, there are ev-

FIG. 1. Gnomonic projection of a normal zone having one principal node at infinity. Between the end-nodes 1, ∞, the derived nodes are spaced according to the terms of the normal complication series N_3.

idently certain preferred arrangements which Goldschmidt has recognized as normal series, N_0, N_1, N_2, N_3, $\cdots$

N_0:	$p=0$								∞
N_1:	$p=0$				1				∞
N_2:	$p=0$		$^1/_2$		1		2		∞
N_3:	$p=0$	$^1/_3$	$^1/_2$	$^2/_3$	1	$^3/_2$	2	3	∞
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

Except in the series N_0, in which there is no development between the end-nodes, 0, ∞, the series show a numerical symmetry about the node 1, which is named the dominant,[4] the terms on the one side being the reciprocals of those on the other. From any series N_n the terms of the next higher series N_{n+1}, are obtained by adding 1 to each term of the series N_n, giving the terms $1 \cdots \infty$ of the series N_{n+1}, and then writing the reciprocals of the terms

[3] V. Goldschmidt, Ueber Entwicklung der Krystallformen: *Z. f. Kryst.*, **xxviii**, pp. 1–35, 414–451, 1896–97. This account is based on these two papers and on conversations with Professor Goldschmidt. The statistical study referred to appears on p. 10.

[4] In the diatonic series, the term 1 corresponds with the dominant of the scale (see below, p. 330).

thus obtained to give the terms $1 \cdots 0$ of the series N_{n+1}. The number of series may be extended indefinitely, but series higher than N_3 are infrequent in crystal zones.

In the belief that these preferred crystallographic number series are of fundamental significance, Professor Goldschmidt has framed a theory of crystal structure and zonal development which leads naturally to the observed preferred arrangements of crystal faces in a zone. This theory rests on two hypotheses:

1. A crystal is a rigid system of like and similarly oriented particles.

2. Every crystal face is a plane normal to a force of attraction exerted from the centre of the crystal particle.

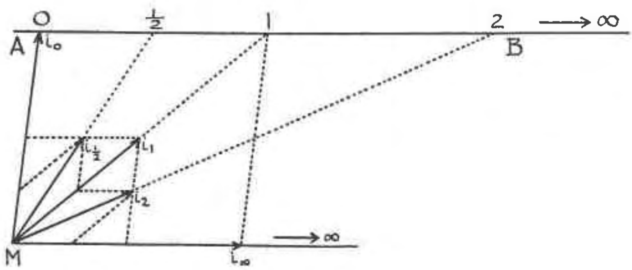


FIG. 2. Section in the plane of a normal zone showing the development of derived forces (and therefore faces) by complication between the principal forces (or faces) i_0, i_∞.

The construction in fig. 2 is a section in the zone-plane of a zone with principal (end) nodes $0, \infty$. AB is the trace of the gnomonic plane and M is the crystal centre. Since 0 and ∞ are principal nodes of the zone, i_0, i_∞ are the principal attraction forces, according to hypothesis 2. The forces i_0, i_∞ are fixed in direction by the nodes $0, \infty$, but their intensities, given by the lengths of the arrows from M, are arbitrary. If the entire forces i_0, i_∞ or aliquot parts, of them, combine to give a resultant by the parallelogram of forces, a new force is produced giving the dominant node 1 and therefore also the corresponding face. Since the principal forces would be exhausted if they combined completely to give a resultant, Goldschmidt assumes that the principal forces divide into aliquot parts, normally in halves, one half of each principal force continuing to act in its respective principal direction and the remaining halves combining to give the resultant force.

$^1/_2 i_0 + ^1/_2 i_\infty = i_1$, by the parallelogram of forces, and therefore, from the initial series:

$$N_0: p = 0 \ \infty$$

we obtain the next higher series:

$$N_1: p = 0 \ 1 \ \infty$$

By combining one half of the remainder of i_0 with one half of i_1, a resultant $i_{\frac{1}{2}}$ is obtained; and similarly one half of i_1 with one half of the remainder of i_∞ gives i_2.

$$^1/_4 i_0 + ^1/_2 i_1 = i_{\frac{1}{2}}; \quad ^1/_2 i_1 + ^1/_4 i_\infty = i_2$$

Whence we obtain the next higher series:

$$N_2: p = 0 \ ^1/_2 \ 1 \ 2 \ \infty$$

Similarly, the successive combination of halves of the remainders of the principal forces with halves of the newly formed resultants give further resultants corresponding to the new terms of the higher series.

To this process of division and combination of parts Professor Goldschmidt has given the name *Complication*, a process which he believes to be fundamental, not only in crystallography, but also in other widely dissimilar fields. In the realm of crystallography complication means specifically the division of the principal crystal forces (giving the principal faces of a zone) into halves, the combination of these parts to give a resultant force (derived face), and the repeated formation of new resultants between adjacent forces by halving and combination of halves.

The empirically found series:

N_0: $p=0$								∞
N_1: $p=0$				1				∞
N_2: $p=0$		$^1/_2$		1		2		∞
N_3: $p=0$	$^1/_3$	$^1/_2$	$^2/_3$	1	$^3/_2$	2	3	∞
. . .	. .	. .	. .	. .	. .	. .	. .	. .

thus becomes the *normal complication series.*

In many zone segments neither of the principal nodes is at infinity. The number series in such zone segments are not normal complication series. If $p_1 \cdots p \cdots p_2$ be such a series, it must be transformed into the standard form $0 \cdots \infty$ by writing $(p-p_1)/(p_2-p)$[5] for each term of the series. This transformation of series into the standard form plays an important part in the critical

[5] For the proof of this formula and the justification for the transformation, see V. Goldschmidt, *Zeit. f. Kryst.*, **xxviii**, pp. 22–23, 1896–97.

discussion of the number series in crystallography and in other fields.

The complication series and the transformation formula for transforming any number series into the standard form $0 \cdots \infty$, give a valuable criterion for the critical discussion of crystal form systems. Uncertain forms may be retained or rejected according to whether they are likely terms of the complication series or not; and well substantiated forms with unusual complication numbers afford evidence of disturbance in the free development of zones.[6]

We are now in a position to state and understand Goldschmidt's general law of crystallography, which may be briefly expressed as follows:

The form system of a crystal results from complication between the principal forces of the crystal particles.

From what has already been given, it is clear that this law accepts and includes, by implication, the Law of Constancy of Angles and the Law of Zones, and recognizes the principal of rationality in the progressive halving of forces which is inherent in the process of normal complication. Goldschmidt's law is more general, however, as we shall see later; at the same time it is more specific in that it recognizes preferred rational number series whereas any simple rational number series would satisfy the older law.

The Law of Complication Applied to Calaverite

According to the principles developed above, the goniometric investigation of a crystal by the Goldschmidt method involves finding the principal nodes and discussing the complication in each zone. The principal nodes are given the simplest possible symbols by suitable choice of axes and unit form, and thus the best elements of the crystal are determined. Then the remaining nodes, which are derived from the principal nodes by complication, are given rational symbols determined by their spacing with respect to the principal nodes. The practical result is the same as that reached by the older method, except that the choice of elements rests on a securer basis since it is made to give the lowest complication between the principal nodes. The theoretical result of the investigation, however, is to prove the validity of the Law of Complication

[6] Outside of Professor Goldschmidt's laboratory, this method of critical discussion of form systems has been usefully applied by W. T. Schaller, The Mercury Minerals from Terlingua, Texas. *U. S. Geol. Sur., Bull.*, **405**, 1909.

as the principal governing the systematic arrangement of crystal faces.

Since the principal nodes could always be regarded as the simplest forms of the crystal, it was natural that the very first step in the Goldschmidt discussion of a projection was to find axes to which the principal nodes were related in a simple manner. But for calaverite this could not be done, and therefore the Goldschmidt method of approach failed, like the others, in the earlier investigations of this mineral. During the last attempt to solve the calaverite puzzle, which at first seemed as hopeless as ever, it occurred to Professor Goldschmidt that it might be better to disregard the question of symbols at first, and to proceed rigorously by finding the principal nodes and discussing the complication between them. This met with immediate success. The entire form system of ninety-two confirmed forms was found to be developed by complication from a few principal nodes; and thus remarkable confirmation of the Law of Complication was obtained in a case where the Law of Simple Rational Intercepts and Indices had failed.

With the best choice of axes, all the principal nodes, except one pair, received the customary simple symbols; the exceptional pair received the exceedingly complicated symbol ($\bar{3}$.5.29), with only tolerable agreement between the calculated and observed angles. It thus became evident that principal nodes might sometimes be forms with complicated, if not quite irrational symbols. Such principal nodes Professor Goldschmidt has proposed to name *singular nodes*.[7] The singular node pair C was the key to the calaverite puzzle. When it was recognised that the node *C* with its complicated symbol was yet the simplest node, namely the node of origin (0), in each zone containing *C*, and that every zone of calaverite gave a more or less complete N_3 complication series without extra terms, it was apparent that calaverite conformed strictly to the Law of Complication as it was formulated many years ago.

The Systematic Crystallography of Calaverite

The practical problem of formally presenting the crystallography of calaverite still remains. The complicated relations are best shown graphically, but for formal purposes they must be expressed quantitatively.

[7] This Journal, **xvi**, p. 78, 1931.

Calaverite is monoclinic and isomorphous with sylvanite. The orthodome zone is developed prismatically; the measurements are therefore projected and discussed in M_2 position (projection plane

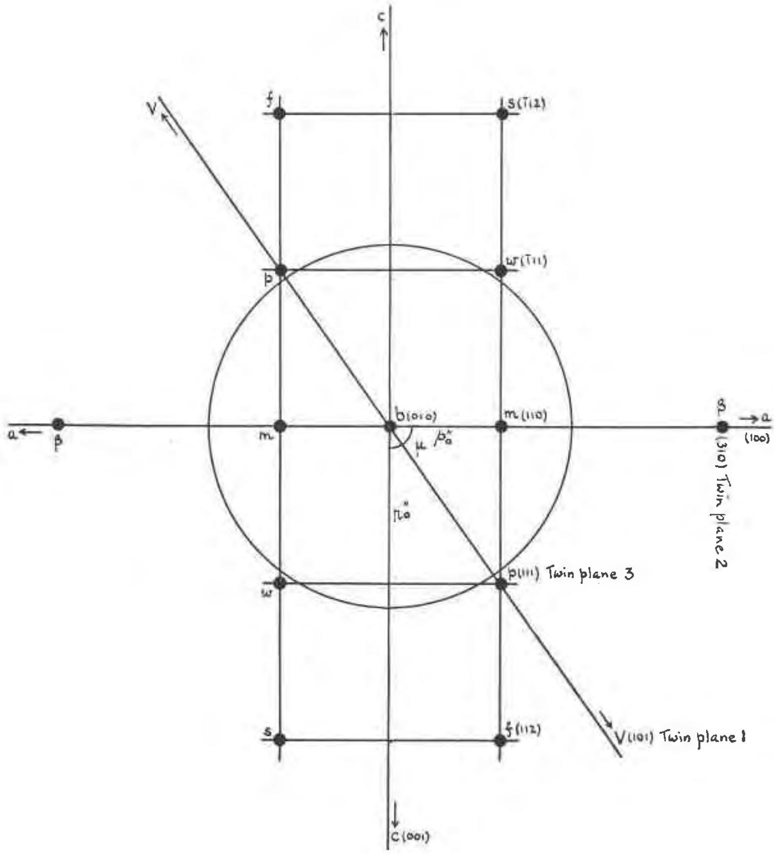


FIG. 3. Calaverite. Gnomonic projection of the S-forms and graphical elements, showing also the three known twinning planes.

normal to the symmetry axis). In this position (fig. 3) the Goldschmidt projection elements are:

$$r_0''=0.8702;\ p_0''=0.6136;\ q_0''=1;\ \mu=89°\ 52'(M_2).$$

The corresponding polar elements in M_1 position (normal position, projection plane parallel to the symmetry axis) are:

$$p_0=0.7051;\ q_0=1.1492;\ r_0=1;\ \mu=89°\ 52'(M_1).$$

These elements are equivalent to the linear elements:

$$a:b:c = 1.6298:1:1.1492;\ \beta = 90°\ 08'.$$

These are the fundamental elements of calaverite; from the isomorphism with sylvanite we have named them the S-elements, and the 10 known forms of calaverite which are simply related to these elements we have named the S-forms.

CALAVERITE—MONOCLINIC
S-ELEMENTS AND FORMS

$r_0'' = 0.8702$; $p_0'' = 0.6136$; $q_0'' = 1$; $\mu = 89°\ 52'$ (M_2)
or $p_0 = 0.7051$; $q_0 = 1.1492$; $r_0 = 1$; $\mu = 89°\ 52'$ (M_1)
or $a:b:c = 1.6298:1:1.1492$; $\beta = 90°\ 08'$

$c(001)$	$a(100)$	$\beta(310)$	$p(111)$	$f(112)$
$b(010)$	$m(110)$	$V(101)$	$w(\bar{1}11)$	$s(\bar{1}12)$

TWINNING LAWS

1. The twinning and composition plane is $V(101)$; the axes of the prismatically developed zones of the two individuals are parallel.

2. The twinning and composition plane is $\beta(310)$; the axes of the prismatically developed zones of the two individuals intersect at 122° 58′.

3. The twinning and composition plane is $p(111)$; the axes of the prismatically developed zones of the two individuals intersect at 93° 40′.

Thus far the statement of the crystallography of calaverite is perfectly normal; but we have accounted for only 10 of the 92 known forms. On the S-elements (M_1) the singular nodes C have the exceedingly complicated symbol $\bar{5}.29.3$, and thus it is clear that, since these are the strongest nodes of the form system, all the forms of calaverite, except the 10 simple S-forms, will receive symbols of great complexity on the S-elements. A table of such symbols would be useless for practical purposes and at variance with the theoretical part of our study.

The only alternative, and it is admittedly a makeshift, is to refer the forms of calaverite to several closely related groups of elements of triclinic character; in this way the great majority of the forms receive simple symbols closely related to their simple complication numbers.

Fig. 4 shows a part of the zone-net of calaverite corresponding to the S-elements (dotted), and in full lines sufficient construction

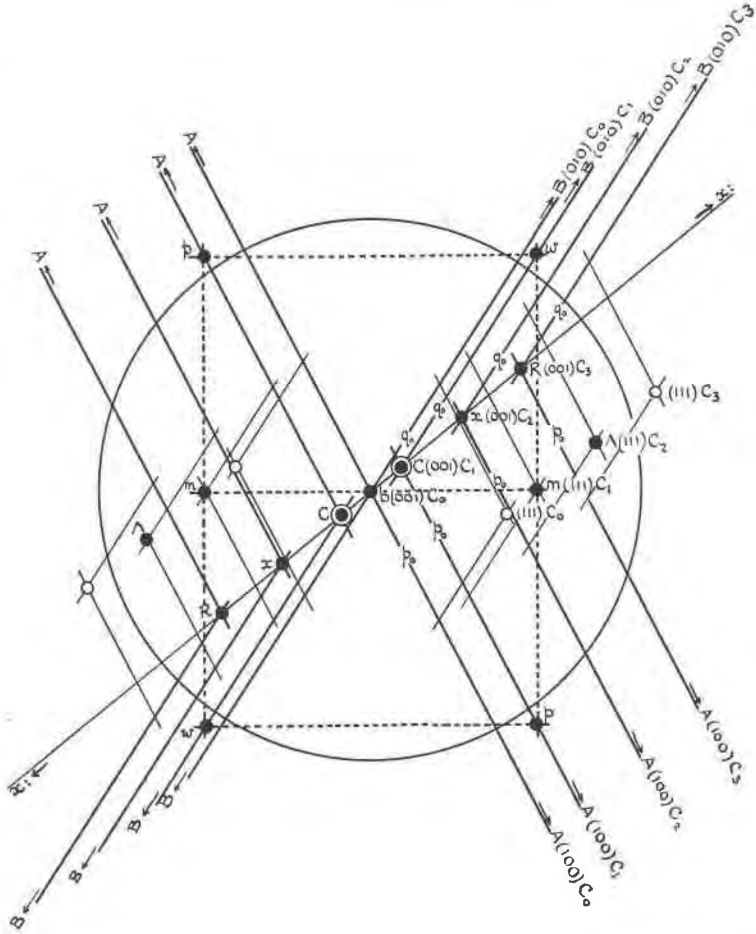


FIG. 4. Calaverite. Gnomonic projection of the C_1, C_2, C_3, C_0 graphical elements (polar axis and unit forms). The principal S-zones are given in broken lines. The blank points are forms not known; the double circles are the singular nodes C. x: is one of the incongrous CC_2-forms lying in zone with C (001) C_1 and a node of the C_2-net.

to illustrate the groups of elements (the calaverite- or C-elements) which serve to accommodate most of the forms of calaverite with simple symbols.

C_1-elements: Polar axes ABC; resp. proj. elements $p_0\, q_0\, r_0 = 1$
C_2-elements: Polar axes ABx; resp. proj. elements $p_0\, q_0\, r_0 = 1$
C_3-elements: Polar axes ABR; resp. proj. elements $p_0\, q_0\, r_0 = 1$
C_0-elements: Polar axes ABb; resp. proj. elements $p_0\, q_0\, r_0 = 1$

These groups of elements have the same polar axes in the plane of the projection, namely AB, and the same projection elements $p_0\, q_0\, r_0$. The remaining polar axes lie in a zone.

It is of no value to express these groups of elements as triclinic linear elements. The linear constants do not show the close relations which exist between the several groups of polar elements, and they would imply a triclinic interpretation of calaverite which we reject.

If calaverite is measured again, it will be in all probability with the 2-circle goniometer. The most useful practical constants to correlate with future work are therefore the 2-circle angles of the polar axes (or primitive pinacoids) ABC, etc., and the projection elements in the plane of the projection, $p_0\, q_0\, r_0$.

Another reason for not recasting the elements in orthodox triclinic form is the fact that we know that even these four sets of axes, or seven sets, when we include the repetition of three of them by the symmetry axis b, are insufficient to account for all the forms. There remain a number of prismatic forms (CC_2-forms) which lie in zone with the base $C(001)$ of the C_1-elements and with nodes of the incongruous C_2-elements. These forms are therefore incongruous to both groups of elements and cannot be given simple symbols.

The following tabulation[8] is therefore offered as the simplest practical quantitative expression of the remarkable form system of calaverite. It will be observed that all the S-forms appear also in the tables of C-forms.[9]

To the reader who may be tempted to measured calaverite on the two-circle goniometer, the following may be useful. Adjust the prismatically developed zone; the face b then lies in the pole and at the centre of the projection. Find the form $m(111)$ C_1, appearing as a pair of excellent faces with polar distance (ρ) $31° 32'$ and with azimuths (ϕ) differing by $180°$. With the azimuth

[8] In principle, this mode of presentation is the same as that adopted by G. F. H. Smith, *Min. Mag.*, **xiii**, p. 125, 1902.

[9] In the joint publication, *N. J. f. Min.*, Beil. Bd. **63A**, 1931, will be found the corresponding measured and calculated two-circle angles, tables 6–10, pp. 30–34.

Calaverite

C_1-Graphical Elements and Forms

$A(100)$	$\phi=62°\ 11'$;	$\rho=90°\ 00'$;	$p_0=0.5419$
$B(010)$	$\phi=\bar{5}7°\ 20'$;	$\rho=90°\ 00'$;	$q_0=0.4646$
$C(001)$	$\phi=\bar{3}8°\ 42'$;	$\rho=\ 8°\ 02'$;	$r_0=1$

$C(001)$	$n(302)$	$w(021)$	$\sigma(131)$	$d(211)$
$B(010)$	$p(201)$	$\xi(031)$	$v(1\bar{1}1)$	$\phi(221)$
$A(100)$	$i(301)$	$r(0\bar{1}1)$	$G(1\bar{2}1)$	$\gamma(231)$
$E(110)$	$L(\bar{1}03)$	$W(0\bar{2}1)$	$\kappa(1\bar{3}1)$	$g(2\bar{1}1)$
$c(1\bar{1}0)$	$Q(\bar{1}02)$	$\Xi(0\bar{3}1)$	$M(\bar{1}11)$	$D(\bar{2}\bar{1}1)$
$V(610)$	$y(\bar{1}01)$	$\tau(122)$	$s(\bar{1}31)$	$\Phi(\bar{2}\bar{2}1)$
$l(103)$	$P(\bar{2}01)$	$\iota(\bar{1}22)$	$t(\bar{1}\bar{1}1)$	$\Gamma(\bar{2}\bar{3}1)$
$q(102)$	$I(\bar{3}01)$	$m(111)$	$\Theta(\bar{1}\bar{2}1)$	$\alpha(321)$
$o(101)$	$u(011)$	$\theta(121)$	$\Sigma(\bar{1}\bar{3}1)$	$\chi(331)$
				$f(3\bar{1}1)$

C_2-Graphical Elements and Forms

$A(100)$	$\phi=62°\ 11'$;	$\rho=90°\ 00'$;	$p_0=0.5419$
$B(010)$	$\phi=\bar{5}7°\ 20'$;	$\rho=90°\ 00'$;	$q_0=0.4646$
$x(001)$	$\phi=\bar{3}8°\ 42'$;	$\rho=23°\ 04'$;	$r_0=1$

$x(001)$	$e(101)$	$\psi(0\bar{1}1)$	$\lambda(\bar{1}\bar{1}1)$	$\delta(\bar{2}\bar{1}1)$
$B(010)$	$\omega(201)$	$k(0\bar{2}1)$	$z(\bar{1}\bar{2}1)$	$H(\bar{2}\bar{2}1)$
$A(100)$	$X(\bar{1}01)$	$\Lambda(111)$	$\Delta(211)$	$h(321)$
$E(110)$	$\Omega(\bar{2}01)$	$Z(121)$	$\rho(221)$	$\beta(331)$
$c(1\bar{1}0)$	$\Psi(011)$	$\Pi(1\bar{1}1)$	$j(2\bar{1}1)$	$F(3\bar{1}1)$
$V(610)$	$K(021)$	$O(1\bar{2}1)$	$U(2\bar{2}1)$	

C_3-Graphical Elements and Forms

$A(100)$	$\phi=62°\ 11'$;	$\rho=90°\ 00'$;	$p_0=0.5419$
$B(010)$	$\phi=\bar{5}7°\ 20'$;	$\rho=90°\ 00'$;	$q_0=0.4646$
$R(001)$	$\phi=\bar{3}8°\ 42'$;	$\rho=35°\ 38'$;	$r_0=1$

$R(001)$	$E(110)$	$S(101)$	$\upsilon(\bar{1}\bar{2}1)$	$N(2\bar{1}\bar{1})$
$B(010)$	$c(1\bar{1}0)$	$\pi(201)$	$T(211)$	$\epsilon(\bar{2}\bar{2}1)$
$A(100)$	$V(610)$	$\eta(0\bar{1}1)$	$\nu(2\bar{2}1)$	

C_0-Graphical Elements and Forms

$A(100)$	$\phi=62° 11'$;	$\rho=90° 00'$;	$p_0=0.5419$
$B(010)$	$\phi=\bar{5}7° 20'$;	$\rho=90° 00'$;	$q_0=0.4646$
$b(001)$	$\phi=-$	$\rho=0$	$r_0=1$

$b(001)$ $B(010)$	$A(100)$ $E(110)$	$c(1\bar{1}0)$ $V(610)$	$J(102)$ $Y(0\bar{2}1)$	$\mu(112)$ $\zeta(212)$

CC_2-Forms

a:	Prismatic form in zone with $C(001)C_1$ and $\rho(221)C_2$
e:	.... $e(101)C_2$
ω:	.... $\omega(201)C_2$
π:	.... $\pi(1\bar{1}1)C_2$
z:	.... $z(\bar{1}\bar{2}1)C_2$
K:	.... $K(021)C_2$
x:	.... $x(001)C_2$
Λ:	.... $\Lambda(111)C_2$

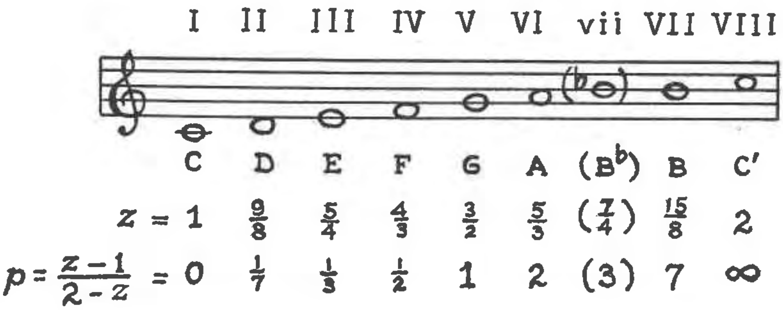
of m as zero meridian the remaining forms will fall into the nets given by the above graphical elements or into the same nets repeated by the symmetry axis b (fig. 4). This agreement will be obtained only if the measured termination is of the same hand as that which we have chosen. If the termination is left handed with respect to our right handed termination (and right and left handed terminations are equally frequent), the measurements will correspond to nets in mirror image relation to fig. 4 (obtained by plotting azimuths in the negative sense). If the termination is a twin after the first law (prismatic axes in common), the observed forms will fall on fourteen nets, the seven shown in fig. 4 and the seven obtained by constructing a similar figure with negative azimuths.

The Law of Complication in Other Fields

In music. More than thirty years ago Professor Goldschmidt showed some striking similarities between the normal crystallographic number series and the series of numbers obtained by suitably transforming the relative vibration periods of the notes of the diatonic scale.[10]

[10] *Ueber Harmonie und Complication*, Berlin, **1901.**

The relative vibration periods (z) of the notes of a major scale, C D E F G A B C′, are very close to the numbers, 1 $^9/_8$ $^5/_4$ $^4/_3$ $^3/_2$ $^5/_3$ $^{15}/_8$ 2, respectively. Transforming this series into the form $0 \cdot \cdot \cdot \infty$, a series of numbers ($p$) similar to a normal complication series is obtained.



Like a normal complication series, this series is symmetrical about the term 1, which corresponds with the fifth or dominant of the diatonic scale. The term (3), corresponding to (B♭), is not in the scale of C major, but is closely related to it musically. Notes with complication numbers of equal rank give pleasing combinations; and the passage obtained by combining notes of equal rank in order of increasing complication, over a sustained bass given by the end terms of the series, is musically perfectly satisfactory.

The terms $^1/_7$, 7, are not found in normal crystal series. In the diatonic series the corresponding notes, D, B, make harsh discords against the end terms, C, C′; but at* in the above passage the effect is inoffensive due to the wide intervals between the discordant notes.

On this foundation Professor Goldschmidt has built up an elaborate system of harmonic analysis.

In colour. The most prominent of the Frauenhofer lines of the solar spectrum, *A* (purple), *B* (scarlet), *C* (red), *D* (yellow), *E* (green), *F* (blue), *G* (indigo), *H* (violet), have the measured wave-lengths (λ) in the following table. The lines cover approximately an octave of the spectrum. Their relative wave-lengths are given very closely by the rational number series, 2 · · · 1, and their relative periods (z) by the series 1 · · · 2. Transforming this series into the form 0 · · · ∞, the series of numbers obtained (p) is again essentially a complication series.

Frauenhofer Line	A	B	C	D	E	F	G	H
λ	7608	6870	6563	5893	5270	4861	4308	3969$\mu\mu$
proportional to.......	1.935	1.748	1.670	1.500	1.341	1.237	1.096	1.010
or nearly............	2.000	1.750	1.667	1.500	1.333	1.250	1.111	1.000
or.................	2	$^7/_4$	$^5/_3$	$^3/_2$	$^4/_3$	$^5/_4$	$^{10}/_9$	1
$z=2/\lambda$	1	$^8/_7$	$^6/_5$	$^4/_3$	$^3/_2$	$^8/_5$	$^9/_5$	2
$p=2(z-1)/(2-z)$	0	$^1/_3$	$^1/_2$	1	2	3	(8)	∞

The series (p) is remarkably like the normal complication series N_3; the term (8) is additional and it corresponds to the colour indigo, which is a colour difficult to distinguish. Except for this term the series is symmetrical about the term 1, which corresponds to yellow, the dominant (brightest) colour in the spectrum. On this basis Professor Goldschmidt develops an extended analogy between sound and colour, and a method for analysing colour combinations.[11]

In cosmic space. The distances of the planets and the asteroids from the sun, reduced so as to make the distance of the earth equal to 10, have long been known to correspond approximately to a number series due to Titius and advocated by Bode.[12] This number series is obtained by writing the series 0, 1, 2, 4, 8, 16, 32, 64, 128, multiplying each term by 3 and adding 4 to each product. The

[11] *Ueber Harmonie und Complication*, p. 83 *et seq.*

[12] H. Shapley and H. E. Howarth, *A Source Book in Astronomy*, New York, **1929**, p. 180.

numbers thus obtained (t) compare with the proportionate mean measured distances (d) as follows:

	Mer.	Ven.	Earth	Mars	Ast.	Jup.	Sat.	Uran.	Nep.
d	3.87	7.23	10.00	15.24	—	52.03	95.39	191.82	300.55
t	4	7	10	16	28	52	100	196	388

Due to the serious discrepancy in the case of Neptune, and the fact that the first term is arbitrary and should really be $(^1/_2 \times 3+4) = 5^1/_2$, this number law is now generally regarded as a curiosity without rational explanation.[13]

In a long paper on Harmony in Cosmic Space Professor Goldschmidt[14] has shown that the relative mean distances of the planets from the sun, transformed, when necessary, by the general method already used in the discussion of crystallographic, diatonic and spectral series, conform to the Law of Complication.

In the following table D is the measured distance of each planet from the sun, in millions of kilometers.

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
D	57.5	107.5	148.7	226.5	773.2	1417.8	2851.3	4467.5
$d=D/733$	0.078	0.147	0.203	0.309	1.055	1.935	3.891	6.096
or nearly..	0.077	0.143	0.200	0.333	1.000	2.000	4.000	6.000
or........	$^1/_{13}$	$^1/_7$	$^1/_5$	$^1/_3$	1	2	4	6

The mean distances are thus very closely represented by the series of rational numbers in the last line of the above table.

The major planets, together with the sun and outer space as end terms, give an incomplete complication series:[15]

	Sun	Jupiter	Saturn	Uranus	Neptune	Outer Space
d	0	1	2	4	6	∞
$p=d/2$	0	$^1/_2$	1	2	3	∞

[13] J. Jeans, *The Universe Around Us*, Cambridge, **1929**, p. 20.

[14] V. Goldshmidt, Ueber Harmonie im Weltraum: *Ann. d. Naturphil.* (Ostwald), **v**, pp. 51–110, 1905.

[15] See Appendix.

Similarly the minor planets, with the sun and Jupiter as end terms, give terms of the complication series:

	Sun	Mercury	Venus	Earth	Mars	Jupiter
d	0	$^1/_{13}$	$^1/_7$	$^1/_5$	$^1/_3$	1
$p=d/(1-d)$	0	$^1/_{12}$	$^1/_6$	$^1/_4$	$^1/_2$	∞
$p'=4p$	0	$^1/_3$	$^2/_3$	1	2	∞
$p''=6p$	0	$^1/_2$	1	$^3/_2$	3	∞

Again, the number series corresponding to the distances of the larger satellites of Jupiter and of Uranus from their respective parent bodies, give incomplete complication series, even without transformation:

	Jupiter	Jo	Europa	Ganymede	Callisto	Outer Space
D	0	419	666	1064	1871	∞
$d=D/970$	0	0.44	0.70	1.11	1.95	∞
p	0	$^1/_2$	$^2/_3$	1	2	∞

	Uranus	Ariel	Umbriel	Titania	Oberon	Outer Space
D	0	205	285	443	583	∞
$d=D/417$	0	0.49	0.68	1.06	1.40	∞
p	0	$^1/_2$	$^2/_3$	1	$^3/_2$	∞

D is the measured mean distance of each satellite in thousands of kilometers.

And in like manner the satellites of Saturn form complication series,[16] and the 700 odd known asteroids are shown to fall into groups concentrated at distances from the sun corresponding to terms of the complication series.[17]

From these facts Professor Goldschmidt develops a cosmogonic theory in which one of the leading thoughts is that groups of bodies forming complication series (or cosmic chords) were each generated in distinct epochs.

[16] V. Goldschmidt, Ueber Harmonie im Weltraum: *Ann. d. Naturphil.* (Ostwald), **v**, p. 77, 1905.

[17] V. Goldschmidt, Harmonie im Reich der Planetoiden: *Ann. d. Naturphil.* (Ostwald), **ix**, pp. 383–392, 1912.

The Significance of the Law of Complication

In the following table we have re-assembled some of the number series found in crystallography, musical harmony, colour and cosmic space. The similarity of the several series is indeed impressive. The examples given, which are the fundamental ones, contain no terms not found in the normal complication series III; the third order of complication seems to be a natural limit which is overstepped only rarely either in crystallography or in the other realms. The collected series show the same order of frequency of terms as that found in the crystallographic series: 0, ∞ (12 times); 1 (10 times); $^1/_2$, 2 (7 to 8 times); $^1/_3$, 3, $^2/_3$, $^3/_2$ (2 to 6 times). In each case the numbers represent lengths or their reciprocals, and where transformation has been necessary to bring the series into the standard form $0 \cdots \infty$, it has been done by a standard operation whose significance in crystallography is understood, implying a change in the position of the projection plane.

Complication Series

Crystallography									
Normal complication series I	$p=0$	·	·	·	1	·	·	·	∞
Normal complication series II	$p=0$	·	$^1/_2$	·	1	·	2	·	∞
Normal complication series III	$p=0$	$^1/_3$	$^1/_2$	$^2/_3$	1	$^3/_2$	2	3	∞
Music									
Major common chord (root position)[18]	$p=0$	$^1/_3$	·	·	1	·	·	·	∞
Major common chord (second inversion)	$p=0$	·	$^1/_2$	·	·	·	2	·	∞
Minor common chord (first inversion)	$p=0$	$^1/_3$	·	·	·	·	2	·	∞
Chord of the dominant seventh	$p=0$	$^1/_3$	·	·	1	·	·	3	∞
Colour									
Purple, scarlet, red, yellow, green, blue, violet	$p=0$	$^1/_3$	$^1/_2$	·	1	·	2	3	∞
Cosmic Space									
Major planets: Sun, Jupiter, Saturn, Uranus, Neptune, Outer Space	$p=0$	·	$^1/_2$	·	1	·	2	3	∞
Minor planets: Sun, Mercury, Venus, Earth, Mars, Jupiter	$p=0$	$^1/_3$	·	$^2/_3$	1	·	2	·	∞
Satellites of Jupiter: Jupiter, Jo, Europa, Ganymede, Callisto, Outer Spac e	$p=0$	·	$^1/_2$	$^2/_3$	1	·	2	·	∞
Satellites of Uranus: Uranus, Ariel, Umbriel, Titania, Oberon, Outer Space	$p=0$	·	$^1/_3$	$^2/_3$	1	$^3/_2$	·	·	∞
Frequency	12	6	7	4	10	2	8	4	12

Carrying the enquiry forward, more by analogy than by direct comparison, Professor Goldschmidt sees the Law of Complication underlying still further groups of natural appearances; among

[18] The chords are here named according to conventional musical theory.

these are certain processes in organic evolution and even some aspects of human physiology and psychology, especially those concerned with the perception of harmonious things. The general conclusion is that the development of the manifold from the simple, by complication, is a universal process. In the realm of sound complication leads literally to an harmonious relation of parts; and therefore, in a more general sense, the universal action of complication points to an harmoniously constructed universe.

The belief in an harmonious universe is not without precedent. Pythagoras knew exactly the numerical relations of the vibration periods of musical intervals. Fully convinced that "number is the essence of things," he dreamed of a universal harmony which could be expressed in rational numbers. With very imperfect knowledge of the solar system, Ptolemy applied himself to finding harmonious relations in the motions of the planets. With much more exact knowledge, Kepler made an extended search for the "harmony of the spheres." This he found, not in the relations of the radii of the orbits of the planets, but in the ratios of their greatest and least apparent angular velocities as seen from the sun, that is in the excentricities of the orbits. Each planet was thus associated with a definite musical interval. A connection between crystals and the planets appears also in Kepler's fanciful disposition of the five perfect Platonic figures (forms in the cubic system) among the orbits of the planets.[19]

In his discovery of a definite number law underlying several extensive and widely different groups of natural phenomena, Goldschmidt has given new significance and precision to the ancient conception of universal harmony. The close approach of the series discussed to an exact mathematical series cannot possibly be the result of coincidence. Adopting the most conservative attitude, we must concede that the discussion points not only to some pervasive principle of rationality underlying the physical phenomena considered, causing certain measures of these phenomena to have simple rational values, but also to a definite preferred order of frequency in these simple rational measures. In the phenomena considered these measures are functions of length; but it is not unlikely, as Goldschmidt believes to be the case in the crystallographic series, that the measures are really functions of force. A similar principle of rationality pervades chemistry, and there is much in the modern

[19] Ioannis Keppleri, *Harmonices Mundi*, Lincii Austriae, v, p. 207, 1619.

outlook in physics to suggest that its fundamentals will eventually be expressed in simple rational numbers.

In crystallography the Law of Complication is evidently a form of the Law of Rationality, and it has proved to be a more general form than the Law of Simple Rational Intercepts and Indices, since it has been found to underlie the hitherto wholly anomalous observations on calaverite. The Law of Complication has furthermore a genetic significance which the older law does not possess; in the hypothesis of crystal development by the interaction of principal crystal forces we have what may be regarded as a first approximation to an explanation of the external form and habit of crystals, an aspect of crystallography—and the one which is after all the most important to the morphologist—which is illuminated neither by the older law nor by röntgenographic studies concerned with the internal arrangement of crystalline matter.

Appendix

Since the above was written, the following publication has been received:

V. Goldschmidt, Der Planet Pluto und die Harmonie der Sphären, *Heidelberger Akten der von Portheim-Stiftung*, **18,** 22 pp., 1932.

In this paper it is shown that the recently discovered planet Pluto, whose orbit has a mean radius of nearly 40 astronomical units, fits squarely into the complication series of the major planets, giving the series:

	Sun	Jupiter	Saturn	Uranus	Neptune	Pluto	Outer Space
$p=$	0	$^1/_2$	1	2	3	4	∞

The new planet thus affords striking proof of the validity of the Law of Complication in the arrangement of the bodies of the solar system.

Professor Goldschmidt briefly reviews his earlier studies of harmonious arrangements in crystallography, music, colour and cosmic space, and gives a generally applicable definition of harmony:

"*Harmony is an arrangement or grouping characterized by the harmonic (complication) series and caused by complication*" (*op. cit.*, p. 17, *trans.*).

The review which we have given is concerned mainly with the numerical basis of Professor Goldschmidt's theory of complication and harmony. To obtain a just conception of the full application and philosophical import of the theory the reader must turn to the original writings. The following list includes those referred to and others not cited:

Victor Goldschmidt (Heidelberg):

1897: Über Entwicklung der Krystallformen, *Zeitschrift für Krystallographie*, **xxviii**, pp. 1–35; 414–451.

1901: *Über Harmonie und Complication*, Berlin.

1904: Über harmonische Analyse von Musikstücken, *Annalen der Naturphilosophie* (Ostwald), **iii**, pp. 449–508.

1905: Beiträge zur Harmonielehre, *Annalen der Naturphilosophie* (Ostwald), **iv**, pp. 417–442.

1906: Über Harmonie im Weltraum, *Annalen der Naturphilosophie* (Ostwald), **v**, pp. 51–110.

1909: Kometen als kosmische Analytiker, *Annalen der Naturphilosophie* (Ostwald), **viii**, pp. 477–482.

1912: Harmonie im Reich der Planetoiden, *Annalen der Naturphilosophie* (Ostwald), **xi**, pp. 383–392.

1919: *Farben in der Kunst*, Heidelberg; second edition, 1929.

1921: Complication und Displication, *Heidelberger Akademie Sitzungsberichte*, A, **12**, pp. 1–90.

1923: *Materialien zur Musiklehre*, vol. **i**, Heidelberg.

1925: *Materialien zur Musiklehre*, vol. **ii**, Heidelberg.

1932: Der Planet Pluto und die Harmonie der Sphären, *Heidelberger Akten der von Portheim-Stiftung*, **18**, 22 pp.