

LATTICE INDICES AND TRANSFORMATIONS IN THE GNOMONIC PROJECTION

M. J. BUERGER, *Massachusetts Institute of
Technology, Cambridge, Massachusetts.*

ABSTRACT

A single, direct, and self-contained demonstration is given of the important self-indexing characteristic of the gnomonic projection. The demonstration is given first for two dimensions, then generalized to three dimensions. The self-indexing characteristic of the projection is limited to the case in which one of the lattice lines chosen as a crystallographic axis is set normal to the plane of projection. The implied lattice significance of changing the origin, length, and positive sense of units is given; these correspond with changing the direction, length and positive sense of the a axis of the lattice, c remaining fixed. Transformation of indices from one plane of projection to another is discussed. This can be resolved into a change of direction of the lattice a axis plus a change of projection from normal to the c axis to normal to the new a axis. The latter transformation requires an index to be replaced by its negative reciprocal. Goldschmidt's transformation:

$$p_1, \dots, p, \dots, p_2 \rightarrow 0, \dots, \frac{p-p_1}{p_2-p}, \dots, \infty$$

is then analysed, and shown to be compounded from the above fundamental transformations.

INTRODUCTION

The use of the gnomonic projection in the selection of a coordinate system for a crystal and for the indexing of crystal faces, is well known to those who have had occasion to study crystal surfaces by ordinary optical reflection methods. A rather considerable background is required however, to satisfy one's self, by the usual formal means,^{1,2} that the projection is self-indexing. This is probably its most important crystallographic characteristic. In the present paper an alternative demonstration is given of the self-indexing property, which is simple and direct. A discussion is also given of the significance of changes in coordinates and of Goldschmidt's transformation.

¹ Boeke, H. E., *Die gnomonische Projektion in ihrer Anwendung auf kristallographische Aufgaben*, pp. 22-26, 1913.

² Palache, Charles. *The gnomonic projection*, *Amer. Mineral.*, vol. 5, pp. 77-79, 1920.

SELF-INDEXING PROPERTY OF THE GNOMONIC PROJECTION

TWO-DIMENSIONAL DEMONSTRATION. It will be most simple to demonstrate the indexing in two dimensions and then show that it can be generalized to three dimensions. Figure 1 shows a general, two-dimensional network of lattice points. The plane of projection, AA , is chosen at right angles to some lattice line BC . A lattice origin is established arbitrarily at B . Through this origin, a normal is constructed to each possible lattice line (equivalent to lattice plane, in three dimensions), BE , BF , BG , etc. The gnomonic pro-

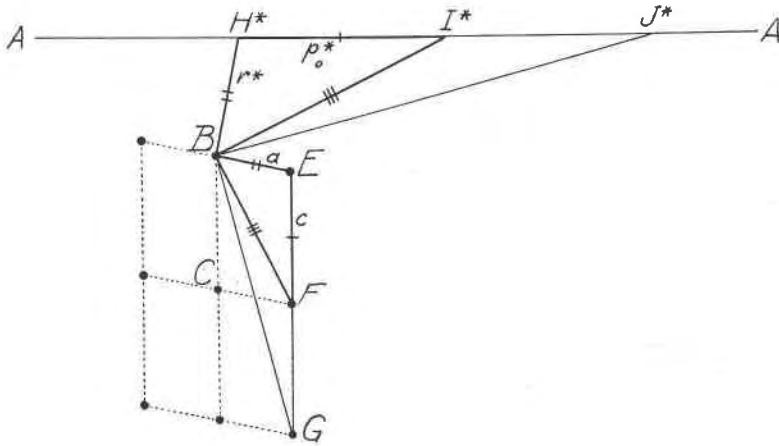


FIG. 1.

jection of the crystal is then the collection of intersections of these normals with the plane of the projection AA .

Suppose $BCEF$ is chosen as the unit cell of the lattice network. The lattice line BE then corresponds to the basal pinacoid, $0(0)1$, the line BF to the unit dome, $1(0)1$, the line BG to the dome, $1(0)\frac{1}{2} = 2(0)1$, etc. (The index number in parenthesis refers to the third lattice axis which does not appear in the present two dimensional demonstration, but which will give convenient comparison in the subsequent generalization to three dimensions.) Triangle BEF has legs at right angles to the homologous parts of triangle BH^*I^* , hence the triangles are similar, and one may write:

$$\frac{H^*I^*}{EF} = \frac{BH^*}{BE} \quad \text{or,} \quad H^*I^* = BH^* \cdot \frac{EF}{BE} = r^* \frac{c}{a}$$

where r^* is the distance of the origin of the gnomonic projection from the lattice origin, and c and a the intercepts of the unit dome on the c and a axes respectively.

For dome $BG = 2(0)1$, triangle BEG is similar to triangle BH^*J^* , from which it follows, as above, that:

$$H^*J^* = r^* \frac{2c}{a}$$

Similar relations hold for other domes. This generalization is conveniently summarized thus:

Plane	Distance from pole of plane to pole of 0(0)1
0(0)1	$r^* \frac{0c}{a} = 0p_0^*$
1(0)1	$r^* \frac{1c}{a} = 1p_0^*$
2(0)1	$r^* \frac{2c}{a} = 2p_0^*$
3(0)1	$r^* \frac{3c}{a} = 3p_0^*$
$p(0)1$	$r^* \frac{pc}{a} = pp_0^*$

where p_0^* stands for the constant r^*c/a , which is fixed by the axial ratio c/a and the scale of the projection, r^* . The reason for the asterisk after certain letters will appear in the generalization to three dimensions. p_0^* is established when the basal pinacoid and unit dome are chosen; it is their distance on the projection. The index number corresponding to the a axis for any other plane is then given by the number of p_0^* units in the distance of its pole from the pole of the basal pinacoid. The resulting index is referred to l is unity, i.e.,

$$p(q)1 = \frac{h}{l} \left(\frac{k}{l} \right) \frac{l}{l}$$

This entire relation is based upon the condition that the plane of projection is normal to the lattice line to be chosen as the c axis.

It is void if this condition is not fulfilled, because the above proof depends upon the similarity of two triangles caused by having homologous legs at right angles. Thus, if the plane of projection is imagined slightly tipped from the position shown in fig. 1, the equal p_0^* spaces between the neighboring faces of the series 0, 1, 2, 3 . . . , change enormously toward the right, scarcely at all near the center of the projection.

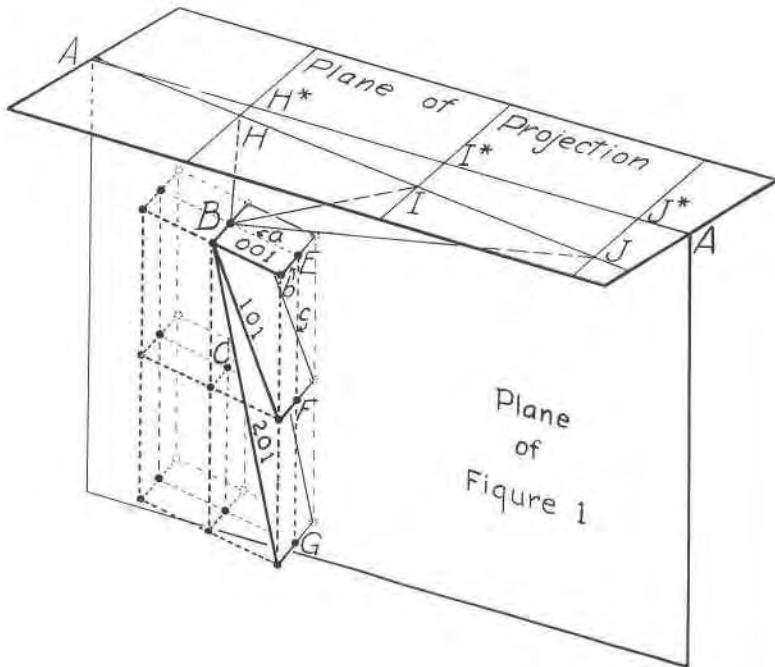


FIG. 2.

THREE-DIMENSIONAL GENERALIZATION. This indexing relationship may now be easily extended to three dimensions. In fig. 2, the plane of fig. 1 is indicated. The lattice is extended beyond this plane in a general manner, after the fashion of a triclinic crystal. The gnomonic projection now consists of the collection of intersections of the normals to lattice planes with the plane of the gnomonic projection. The normal to the basal pinacoid is no longer in the plane of fig. 1, but rather at H . The zone containing the domes, 101, 201, 301 . . . 100, corresponding to the zone discussed for two dimensions, is normal to 100 and $\bar{1}00$, i.e., normal to one of

the sides of the lattice cell, and through the point $H(=001)$. There is a corresponding zone through H running at right angles to the other vertical cell side, 010. These two zones may be regarded as fundamental lines indicating the directions of coordinate axes in the projection.

The locus of the projection of all planes containing the line BE is the last-mentioned zone line, HH^* . In the special case of a plane in this zone normal to the plane of fig. 1, the normal corresponds to the spot H^* , the projection of the basal pinacoid in the two-dimensional discussion. Likewise, the locus of the projection of all planes containing the line BF is a zone line II^* ; in the special case

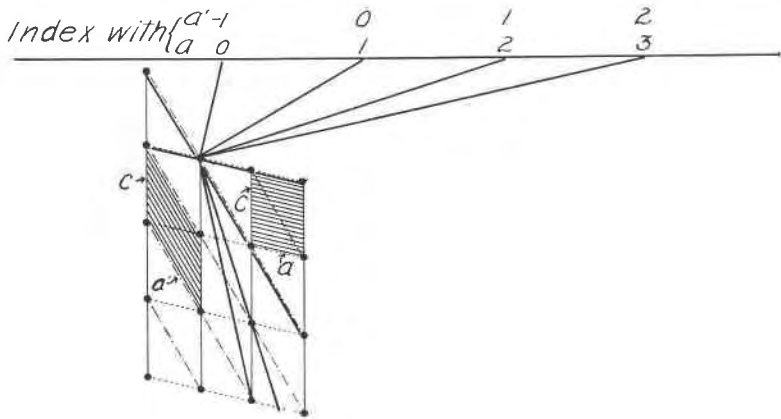


FIG. 3.

of a plane in this zone normal to the plane of fig. 1, the projection point is I^* . Similarly, the projection of any dome is in the same zone as the projection of the corresponding two-dimensional dome. The zone lines just mentioned are all parallel, because they contain the pole of 010 in common, i.e., they are all normal to the 010 plane of the lattice. They are also equally spaced, because they divide the line $H^*I^*J^* \dots$ into equal p_0^* intercepts. It follows that they also divide the line $HIIJ \dots$ into equal p_0 intercepts, and that p_0^* and p_0 are related to one another by a proportionality constant. It is not necessary to derive this constant here, the point of the demonstration being that the gnomonic projection is self-indexing in three dimensions just as in two dimensions, once a basal pinacoid and unit dome are selected. The indexing in the remaining dimension follows from a discussion similar to the above.

TRANSFORMATIONS

POSSIBLE FUNDAMENTAL CHANGES. It follows from the preceding discussion that, for a given unit cell, only one indexing of the projection is possible. From a projection point of view, the indexing of a fundamental line (zone) is also fixed if a choice is made of origin, length of unit, and positive direction of unit. It follows that a given choice of these three elements implies a distinct unit cell. The changes in the implied cell for changes in choice of these ele-

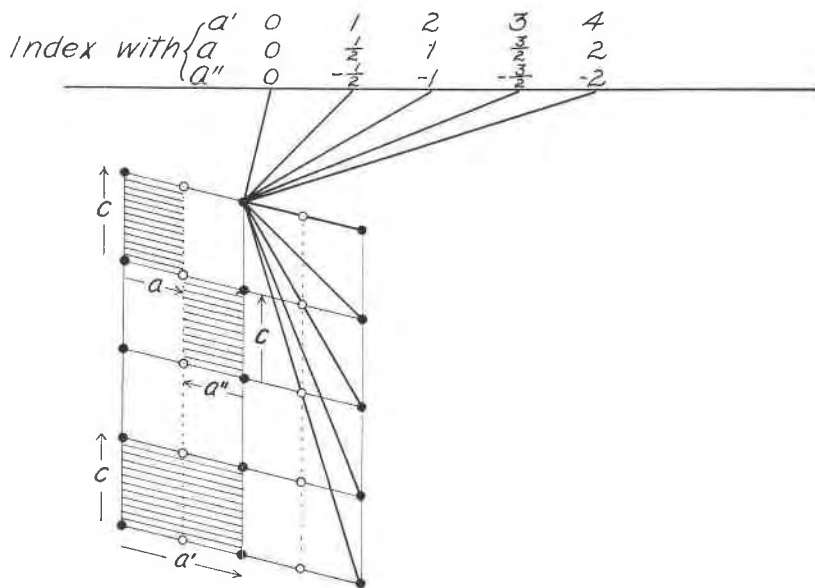


FIG. 4.

ments is most simply followed in a single projection line, corresponding to a two dimensional lattice:

CHANGE OF ORIGIN. Fig. 3 indicates that a shift of origin to +1 (i.e., index $p \rightarrow p' - 1$) is equivalent to changing the direction of the a axis of the unit cell so that it is parallel to the direction of the original unit dome. In general, a change of projection origin so that $p \rightarrow p' - n$ implies a shift of a axis direction such that the end of the a translation takes a new position, $-n$ lattice points along the c direction (i.e., n lattice points along the $-c$ direction), the lattice origin remaining undisturbed.

CHANGE OF UNIT. Fig. 4 indicates that halving the projection unit, which is equivalent to doubling the index of any pole, implies a doubling of the a axis length of the unit cell, the direction of the axis remaining constant. In general, multiplying the designation of a pole by $+m$ implies multiplying the length of the lattice a axis by $+m$. Alternatively, it implies dividing the length of the c axis by $+m$. In dealing with projections alone, where the absolute a and c lengths of the lattice cannot be dealt with, it is

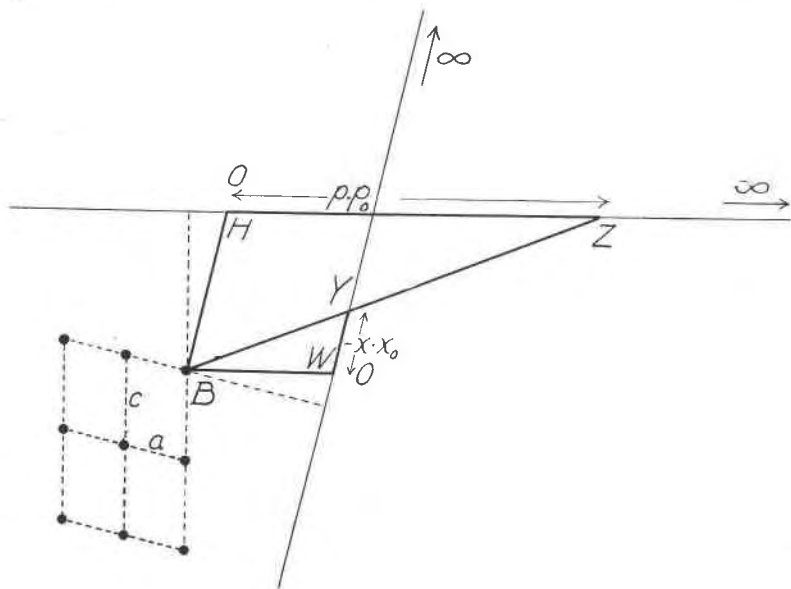


FIG. 5.

better stated that multiplying the designation of a pole by $+m$ implies multiplying the axial ratio a/c by $+m$. If c is arbitrarily regarded as fixed, this statement reduces to the first.

CHANGE OF POSITIVE DIRECTION. Fig. 4 indicates that multiplying the designation of a pole by -1 is equivalent to choosing a new a axis in the original $-a$ direction.

It has already been shown that the self indexing property of the gnomonic projection depends on having one axis (here designated the c axis) normal to the plane of projection. The c axis direction thus fixed, only the length of the c axes, the direction (including

sense) of the a axis, and the length of the a axis remain as possible variables. All of these are included in the above transformations.

CHANGE OF PLANE OF PROJECTION. One other transformation is of importance, namely the change in indexing when the plane of projection is shifted from normal to one lattice line to normal any other desired lattice line. The first lattice line is the c axis according to the designations followed above. The second lattice line is the a axis, or it may be made to become the a axis by the simple expedient of changing the original direction of the a axis to the desired lattice line. This is accomplished by first changing the projection origin through addition of a constant to all indices, as indicated above.

Fundamentally, we require to know how the index, p , of a pole transforms when the plane of projection is changed from normal to the c axis to normal to the a axis, the same unit cell being retained. This can be derived from fig. 5. The index of the pole on the new projection normal to the a axis is designated $-x$. The origins and desired pole are indicated on both projections. In the figure, HZ and BW are parallel, and BH and WY are parallel, because these pairs are normal to pinacoid planes respectively. Also, BY and BZ are identical. Hence triangles YWB and BHZ are similar. This gives the proportion,

$$\frac{-x \cdot x_0}{BW} = \frac{BH}{p \cdot p_0} \tag{1}$$

from which

$$x \cdot x_0 = - \frac{1}{p \cdot p_0} (BW \cdot BH) \tag{2}$$

and

$$x = \frac{1}{p} \frac{(BW \cdot BH)}{x_0 \cdot p_0} \tag{3}$$

The term in parenthesis is an area dependent upon choice of distances between lattice origin and projection origins, and hence is constant for the transformation. In absolute units, $p p_0$ transforms directly into its negative reciprocal provided $(BW \cdot BH)$ is chosen equal to unity. For the purpose of indexing, this particular choice of scales is of no importance, for each projection may have its own

scale of x_0 and p_0 units, such that $x_0 = (BW \cdot BH)/p_0$. This makes the last term of the left hand member of (3) equal unity, and p transforms directly into its negative reciprocal. In practice, this is easily arranged by noting that the pole unity transforms into negative unity.

GOLDSCHMIDT'S TRANSFORMATION. In applying the law of complications, Goldschmidt³ found it occasionally desirable to transform the series of indices, $p_1 \cdots p \cdots p_2$, occurring in a zone sequence, to the form $0 \cdots \infty$. This is done by multiplying each of the terms, p , of the series by the constant $p - p_1/p_2 - p$. Goldschmidt only partly justified this transformation, and in any case, the physical significance of each operation of the transformation is lacking. From the transformations just established, Goldschmidt's transformation may be analysed as follows:

(1) To each term, $p_1 \cdots p \cdots p_2$, is added the constant, $-p_2$, giving the series, $p_1 - p_2 \cdots p - p_2 \cdots 0$. Goldschmidt remarks that this is equivalent to shifting the origin of the series. The more fundamental significance is evidently that it is equivalent to choosing a new unit cell of the same volume as the original, but with a different particular position of the cell base. The addition of the particular constant, $-p_2$, to each term, makes the dome corresponding to the original p_2 become the basal pinacoid, with a pole at 0. This transformation is preliminary to changing the plane of projection.

(2) Each of the above terms is inverted giving the series $1/p_1 - p_2 \cdots 1/p - p_2 \cdots \infty$. Goldschmidt justified this on the ground that a series, $0 \cdots \infty$, has the same significance as the series $\infty \cdots 0$. This is not a very satisfying reason. Fundamentally, inversion is equivalent to a permutation of negative a and positive c axial directions in the lattice, which is the same as a shift of the plane of projection from a position normal to the c axis to one normal to the a axis, plus an interchange of positive and negative a axis directions, as shown in a preceding paragraph.

(3) The series is then multiplied by a constant, $p_1 - p_2$, giving the series $1 \cdots p_1 - p_2/p - p_2 \cdots \infty$. Goldschmidt justified this by saying that multiplication by a constant means altering the choice

³ Goldschmidt, V., Ueber Entwicklung der Krystallformen: *Zeit. Krist.*, vol. 28, 1897, pp. 1-35 and 414-451, especially pp. 22-23.

of force units to change the length of the zone. This explanation is, unfortunately, bound up with Goldschmidt's interpretation of the significance of the law of complications. As already demonstrated, the lattice significance of multiplying the designations of the nodes by a constant, $+m$, is that a new unit cell is chosen having the length of the a axis $+m$ times that of the old a axis. In particular, multiplication by $p_1 - p_2$ is equivalent to lengthening the new a axis to stretch the zone from unity to infinity. The last term is already ∞ ; this operation specifically changes the first term to unity.

(4) Finally, a constant 1 is subtracted from each term in order to make the first term zero. The lattice significance of this is the same as in (1), above, namely, a shift of the a axis direction.

Goldschmidt's transformation may be summarized as follows: Any segment of a zone may be stretched to cover the gnomonic projection range 0 to ∞ by an appropriate choice of the plane of projection and an appropriate choice of axes for the unit cell of the lattice. Specifically, a new plane of projection is sought parallel to the normal to one of the two terminal faces of the zone segment; this makes the index of this pole become infinity on the new projection. The other terminal face is made to have the desired index, 0, by multiplying the entire series by a constant, followed by adding a constant; this is equivalent to choosing the new a axis (old c axis) with a new length and a new direction.