A DEFINITION OF EUCLIDEAN GEOMETRICAL SYMMETRY*

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ABSTRACT

A predominantly verbal definition of Euclidean geometrical symmetry is given in the form of a series of definitions, which are briefly discussed. Examples of non-operational statements corresponding to symmetry operations are formulated. Sufficiency of the definition is not proved.

An understanding of the concepts of geometrical symmetry is of value to most scientists and of prime importance to mineralogists and physicists. Yet the student seeking a definition of geometrical symmetry most likely faces disappointment, particularly if he consults textbooks of mineralogy. There he will find either no definition at all or verbal ones which are at best vague and insufficient, at worst misleading. College students usually have a fair knowledge of Euclidean geometry, so there is no reason why they should not be given an adequate definition in terms with which they are familiar.

The following definition of symmetry is given in the form of a series of definitions in an effort to satisfy the mandate of unequivocal statement.

(1) A configuration is any collection of points in an Euclidean medium. (Line, plane or space.)

(2) One-to-one correspondence between the points of two configurations C and C′ exists when each point of C′ corresponds to a single point of C and no two points of C′ correspond to the same point of C. (C and C′ then have the same number of points.)

(3) A PrP′-relation is any relation defining a one-to-one correspondence between each point P of a configuration and a point P′ of the same configuration. (The configurations C and C′ are identical.)

(4) A PrP′-relation is a symmetry-relation when the following three conditions are fulfilled:

(*a*) The distance between any two points P_1 and P_2 is equal to the distance between the corresponding points P_1' and P_2'.

(*b*) At least one point P is not identical with its corresponding point P′.

(*c*) Every point P is also a point P′.

(5) A configuration possesses the property of symmetry when it validates one or more symmetry relations.

The first of the three conditions that make a PrP′-relation a symmetry relation needs no comment. The second condition excludes all generally

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valid relations, i.e. relations which do not give any significant information. Thus the covering operation of rotation through 360° is not a symmetry operation, the plane containing all the points of a configuration is not a mirror plane, and the line containing all the points of a configuration is not an axis of rotational symmetry.

The third condition may be somewhat less easy to understand. It implies that every point can occur "at both ends" of a symmetry relation. Consider for instance the PrP′-relation "P′ lies one inch to the right of P." Being a member of the configuration, P′ is of course also a point P so that there must be a third point one inch to the right of P′, and so forth. We end up with a row of equally spaced points, starting at P and extending to infinity in a direction to our right. In this case our PrP′-relation is not a symmetry relation although it satisfies the first and second conditions. It is, however, a symmetry relation (translation) if the row of points also extends to infinity on our left. Now this extension is called for by the third condition, for if P is also a point P′, then it lies one inch to the right of another point of the configuration, which of course is located to the left of P, and so on.

We can now easily formulate the symmetry relations corresponding to the simple operations of translation, rotation, reflection, and inversion.

(1) The vector P→P′ is parallel and equal to a given vector (direction and period of translation.)

(2) P and P′ lie on a circle normal to and centered on a given line (axis of symmetry), and the angular rotation from P to P′ has a given value (angle of rotation) and a given sense (clockwise or counter-clockwise).

(3) The line through P and P′ is normal to a given plane (plane of symmetry) and the point of intersection with the plane bisects the line segment PP′.

(4) A given point (center of symmetry) bisects the line segment PP′.

Reflection and inversion are thus explicitly defined in non-operational terms, but the same is not true of translation and rotation. The bivectorial nature of translation is only implied by the above definition, as is also the irrelevance of the sense of rotation. Explicit definitions of translational and rotational symmetry follow.

(5) The line through P and P′ is parallel to a given line (direction of translation) and the distance between P and P′ has a given value (period of translation).

(6) P and P′ lie on a circle normal to and centered on a given line (axis of symmetry), and the angular distance between P and P′ has a given value (angle of rotation).

These satisfactory definitions can, however, not be given in the first place because they reveal a one-to-two correspondence which is not explicit in the stated definition of symmetry, but which can be deduced from it.

Complex symmetry relations (corresponding to translatory-rotation, translatory-reflection, rotary-reflection, and rotary-inversion) can be defined as follows:

First simple relation:	P′ corresponds to P.
Second simple relation:	P″ corresponds to P′.
Complex relation:	P″ corresponds to P.

The point P′ is not necessarily a member of the configuration.

Equivalence of points is also easy to define: Corresponding points are equivalent. Furthermore if P′ corresponds to P by virtue of a symmetry relation and P″ corresponds to P′ by virtue of the same or a different relation, then P and P″ are also equivalent.

An appraisal of the sufficiency of the definition of symmetry offered above indicates that it covers all the symmetries within the grasp of an Euclidean imagination, i. e. symmetries truly representable by means of points which have the single property of position. Symmetries only visualizable as properties of physical phenomena can be classed as not strictly Euclidean and are excluded by our definition. The holohedral spherical point group K_h, for instance, is covered by our definition while the enantiomorphous spherical group K is excluded. It also follows that geometrically C_∞ is indistinguishable from $C_{\infty v}$ and that $C_{\infty h}$ and D_∞ are equal to $D_{\infty h}$. Applied to continuous and semi-continuous symmetry groups, this criterion admits the second order groups as defined by Heesch[1], but excludes many of the symmetries described by Schubnikow[2].

In conclusion it is admitted that it remains to be proved that no asymmetric relation will satisfy the definition. A modification may become necessary to exclude asymmetric relations which have not occurred to the author.

[1] Heesch, H., Ueber die Symmetrien zweiter Art in Kontinuen und Semidiskontinuen: *Zeit. Krist.*, **73** (1930).

[2] Schubnikow, A., Ueber die Symmetrie des Kontinuums: *Zeit. Krist.*, **72** (1929).
Ueber die Symmetrie des Semikontinuums: *Zeit. Krist.*, **73** (1930).