CHARTING FIVE AND SIX VARIABLES ON THE BOUNDING TETRAHEDRA OF HYPERTETRAHEDRA*

JOHN B. MERTIE, JR.

CONTENTS

ABSTRACT

Triangles, tetrahedra, and hypertetrahedra may be used as reference frames for charting variables whose sums equal unity. The number of variables that may theoretically be charted equals the number of vertices in the figure.

Hypertetrahedra of n dimensions are bounded by vertices, edges, triangular faces, tetrahedra, and hypertetrahedra of n-1 and fewer dimensions. Direct geometric charting with regard to hypertetrahedra is impossible, as such figures can not be envisaged or constructed; but the variables may be plotted in groups in relation to the bounding triangles and tetrahedra.

Methods were recently presented by the writer for charting five, six, and seven variables on the bounding faces of hypertetrahedra of four, five, and six dimensions. Methods are now presented for charting five and six variables in relation to the bounding tetrahedra of hypertetrahedra of four and five dimensions. For seven variables, these methods have no advantage over charting on the bounding triangular faces. The plotting of variables in relation to the bounding tetrahedra produces surfaces or space curves, of which contour maps or calibrated plane curves are made. Charts are given for the proper presentation of such figures.

Another method is given for projecting directly the quintets of a hypertetrahedron of four dimensions into relationship with one of its bounding tetrahedra. This results in the production of surface contour maps, which ordinarily are best presented as models, photographs, or perspectives. This method is also generalized into five dimensions, but the results are regarded as impracticable.

Trilinear and quadriplanar coordinates are used in this presentation; and data are given for the use of negative coordinates, and for magnification of scale.

INTRODUCTION

Variables or components, whose sums total 100 per cent, may be charted on the boundaries of hypertetrahedra of n dimensions. Such boundaries comprise vertices, edges, triangular faces, tetrahedra, and hypertetrahedra of less than n dimensions. The boundaries that are useful in practical charting are the triangular faces and the tetrahedra. In a paper recently published, the writer† showed how the bounding triangu-

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† Mertie, John B., Jr., Charting five, six, and seven variables on hypertetrahedral faces: *Am. Mineral.*, **33**, 324–336 (1948).

lar faces may be utilized in charting five, six, and seven variables. The present paper, which is a continuation and completion of this topic, presents methods for charting five and six variables in reference to the bounding tetrahedra.

Trilinear Coordinates

The graphic utilization of positive trilinear and quadriplanar coordinates is almost universal among research workers in the physical sciences, so that this usage need not be described. The analytic applications of these coordinates are not at all well known, but inasmuch as this topic is not a part of this paper, it will likewise be omitted. Certain graphic applications, however, that are not well known, are worthy of mention.

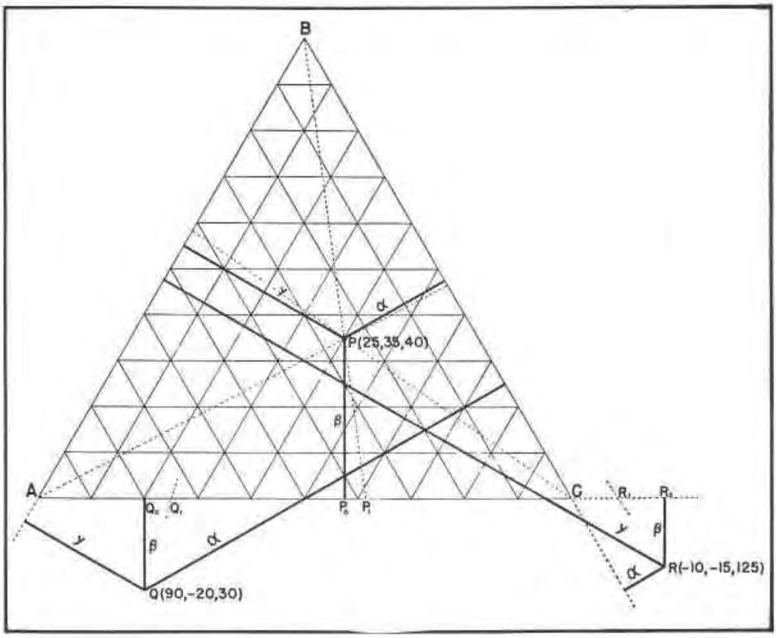

Fig. 1. Positive and negative trilinear coordinates.

An equilateral triangle of reference, ABC, known as a trigon, is shown in Fig. 1. A point P, having the trilinear coordinates (α, β, γ), or specifically (25, 35, 40), is also shown. Draw the line BP and extend it to meet the side AC. The intersection, P_1, defines the point that will result from charting the components A and C, if these are recomputed to 100 per cent. Similar points are indicated, though not lettered, on the sides AB and

BC. The point P_0 is the orthogonal projection of P on AC. The bilinear coordinates of P_0 on the line AC are found to be $(\alpha+\beta/2)$, $(\gamma+\beta/2)$, or specifically $(42\frac{1}{2}, 57\frac{1}{2})$. The orthogonal projections of P on the sides AB and BC will define two other points similar to P_0, whose γ and α coordinates are respectively eliminated. This simple projective relationship may be generalized to three and four dimensions.

Negative trilinear coordinates may also be charted. One example of their application was given in the paper cited above, and others might be mentioned. Trilinear coordinates may comprise three positive values, two positive and one negative values, or one positive and two negative values. Three negative coordinates are not possible. A set of coordinates with one negative value will define a point outside the trigon, and opposite the side from which the negative coordinate is measured. A set having two negative coordinates will define a point within the exterior angle formed by extending the sides from which the negative coordinates are measured. Examples in Fig. 1 are the points Q and R, having respectively the coordinates (90, −20, 30) and (−10, −15, 125). The point Q_1 is defined exactly as was the point P_1; and giving due regard to the negative value of β, the point Q_0 becomes the orthogonal projection of Q on the side AC. Using the negative values of α and β, of the point R, the points R_1 and R_0 are similarly located.

Quadriplanar Coordinates

A regular tetrahedron of reference $ABCD$, with the front face ADC removed, is shown in perspective in Fig. 2. The bounding faces are lined for trilinear coordinates, so that ABC, ABD, and BCD are triangles of reference, or trigons, similar to the one shown in Fig. 1. The point P has the quadriplanar coordinates $(\alpha, \beta, \gamma, \delta)$, or specifically (15, 20, 30, 35). Draw the line DP and extend it to intersect the face ABC. The resulting point, P_1, has the trilinear coordinates that will be obtained by recomputing to 100 per cent the components A, B, and C. The point P_0, which is the orthogonal projection of P onto the side ABC, will be found to have the trilinear coordinates $(\alpha+\delta/3)$, $(\beta+\delta/3)$, $(\gamma+\delta/3)$, or specifically $(26\frac{2}{3}, 31\frac{2}{3}, 41\frac{2}{3})$. Points similar to P_1 and P_0 may be located on each of the other three faces of the tetrahedron, and are in fact shown as unlettered small circles. It is thus feasible to project readily a surface, defined by sets of quadriplanar coordinates, onto any face of a tetrahedron, though ordinarily one such projection will suffice. A line contour map may then be made by drawing curved lines through or between the projected points, at intervals corresponding to selected values of δ, the eliminated coordinate.

Negative quadriplanar coordinates may also be charted, and one example of this application was given in the preceding paper. A set of quadriplanar coordinates may comprise four positive values, three positive and one negative values, two positive and two negative values, or one positive and three negative values. Four negative values are not possible. A set with one negative value will define a point outside the

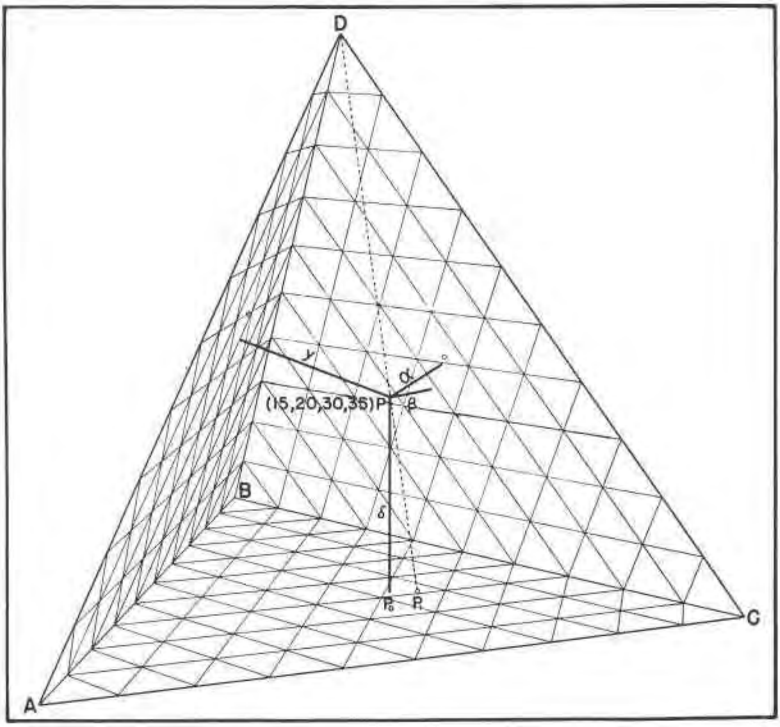

FIG. 2. Quadriplanar coordinates within a tetrahedron.

tetrahedron of reference, and opposite the face from which the negative coordinate is measured. A set with two negative values will define a point within the exterior dihedral angle formed by producing the two faces from which the negative coordinates are measured. A set with three negative values will define a point within the exterior trihedral angle formed by producing the three faces from which the negative coordinates are measured. Giving due regard to the negative values of the coordinates, points similar to P_1 and P_0 may be located on all of the bounding faces, or on their extensions beyond the tetrahedron of reference. Surfaces lying inside or outside the tetrahedron of reference may then be shown as line contour maps, either on the faces of the tetrahedron or on their extensions.

Quartets that are plotted as points in relation to a tetrahedron of reference theoretically define a surface, of which a line contour map may be made. But if such points tend to have a linear disposition, they may be better treated by joining them together as a space curve, and by projecting the space curve orthogonally as a plane curve onto one face of the tetrahedron. The altitudes, or values of the eliminated coordinate, may then be used for calibrating the plane curve, so that it will finally have the appearance of a calibrated curved scale in a nomogram. If some meaning is attached to the sequence of points that lie on the space curve, an arrow may be used on the projected plane curve to indicate this sequence; and if the space curve is numerically related to some physical property, such as grain size, specific gravity, or magnetic susceptibility, the projected plane curve may be doubly calibrated. Thus one calibration will refer to the eliminated coordinate, and the other will give the values of the physical property that is to be shown. Physical properties may similarly be represented on the contour map of a surface, producing thereby a grid pattern. Surfaces and space curves on such surfaces may thus be simultaneously projected, with the result that one or more lines (calibrated or uncalibrated) will intersect the contour lines.

Magnification of Scale

The topic of scale, in plotting triads in relation to trigons, was discussed in the preceding paper.* The methods there given may readily be applied to three dimensions, for plotting quartets in relation to tetrahedra of reference. For magnification of scale, one merely imagines that the tetrahedron is larger than it really is, so sets of coordinates are plotted farther from the vertices of reference than they would normally be. The ratio of the altitude of the imaginary to that of the real tetrahedron of reference is the magnification of scale.

The determining factor in the enlargement of scale is the minimum value of the largest coordinate of a number of sets that are to be charted. Thus if the smallest value of some one coordinate, say α, that exceeds all others in a number of sets, is 50, the maximum amplification of scale is obviously 2. Similarly, if this minimum value of α is 90, the maximum amplification of scale is 10. The same table of possible magnifications that was given in the preceding paper therefore holds for tetrahedra of reference as well as for trigons, as do also the simple formulae earlier

* Mertie, John B., Jr., *Op. cit.*, pp. 332–333.

given. One statement in the earlier paper, however, needs clarification. On page 333 it was stated that "it is immaterial whether the largest coordinate in the sets is α, β, or γ, *or a mixture of these.*" Insofar as any one trigon is concerned, the italicized clause is incorrect and should be deleted. The clause does apply, however, to different trigons in a composite chart.

The magnification of scale in charting triads was shown as a number at the orthocenter of each trigon. For triangles representing the bases of tetrahedra, this central position is preempted by another number that identifies the projected vertex. Some convention must therefore be devised to avoid confusion between these two sets of numbers. It is recommended that the scale factor, both for trigons and for tetrahedra of reference, be placed inside the figure, at the apex that corresponds to the maximum coordinate of the sets. With one exception, the scale number will thus appear in the corners of the triangles. If the largest quadriplanar coordinate of a number of sets, however, is that of the projected apex, the scale number and the apex number must appear together. It is recommended, for this exceptional case, that the scale number be given as a subscript of the apex number, as for example, 5_{20}. These placements of the scale number will serve two purposes: first they will indicate the degree of magnification, and second, they will show the direction of elongation of the imaginary trigon (or tetrahedron of reference).

Hypertetrahedral Charting

Points that define a surface referred to a tetrahedron may be charted either in multiple on the bounding faces of the tetrahedron, or directly by orthogonal projection onto one of these faces. These methods correspond to plotting four variables, either as four sets of triads defined by points similar to P_1, of Fig. 2; or as one set of points similar to P_0, from which a line contour map may be made. The same alternatives exist in charting points having five coordinates which define a continuum referred to a hypertetrahedron of four dimensions; but the latter is bounded by tetrahedra as well as triangular faces. Therefore, for the hypertetrahedron of four dimensions, three kinds of charting may be done; first, the five variables may be recomputed as triads and charted on or with relation to the bounding triangular faces; second, the five variables may be recomputed as quartets and charted in relation to the bounding tetrahedra; and third, the four-dimensional points may be projected directly into or with relation to a single bounding tetrahedron. The first of these methods was described in the writer's first paper on this subject. The second, which is also a practical method, is described below. The third, though rather difficult and less practicable, is also outlined here.

The most practical methods, either for five or six variables, are to recompute the components as triads and chart them in trilinear coordinates, or to recompute the components as quartets and chart them in quadriplanar coordinates. These two methods require a knowledge of the number of triangles and tetrahedra that bound the various hypertetrahedra. The number and character of these boundaries, though given in the preceding paper, are reproduced again for reference.

BOUNDARIES OF HYPERTETRAHEDRA, FROM FOURTH TO NINTH DIMENSION

	Fourth	Fifth	Sixth	Seventh	Eighth	Ninth
Vertices	5	6	7	8	9	10
Edges	10	15	21	28	36	45
Triangles	10	20	35	56	84	120
Tetrahedra	5	15	35	70	126	210
H_4	1	6	21	56	126	252
H_5	0	1	7	28	84	210
H_6	0	0	1	8	36	120
H_7	0	0	0	1	9	45
H_8	0	0	0	0	1	10
H_9	0	0	0	0	0	1

In this tabulation, H_4, H_5, etc., refer to hypertetrahedra of the fourth, fifth, and higher dimensions.

This tabulation shows that 10 triangles or 5 tetrahedra are required for charting 5 variables; that 20 triangles or 15 tetrahedra are required for charting 6 variables; and that 35 triangles or the same number of tetrahedra are required for charting 7 variables. The charting of quartets referred to a tetrahedron is superior to that of charting triads referred to a trigon, because the relationships between four variables are simultaneously shown, but the former method is more laborious than the latter. For five or six variables, however, the additional labor is to some degree compensated by the fact that fewer tetrahedra than triangles are required. But for seven variables, no such compensation exists, as the number of required tetrahedra and triangles are the same, and charting on the bounding tetrahedra becomes very laborious. Therefore the method of charting quartets in reference to bounding tetrahedra is utilized only in plotting five and six variables.

When five or six variables are recomputed as quartets, and are plotted with reference to bounding tetrahedra, the resulting surfaces may be shown either as three-dimensional models, as photographs or perspective drawings, or as line contour maps. Space curves may be similarly shown, except that one or more calibrated plane curves will result instead of a line contour map. The method of line contour mapping and of space curve projection is here utilized. Five tetrahedra are required to chart the five variables, and therefore one face of each tetrahedron needs to be reproduced in order to show five contour maps or plane curves. This requires a compact arrangement of five triangles. Four such arrangements are shown in Fig. 3, but obviously the one in the upper left of the drawing is the most compact and therefore the best. As each of these triangles represents one face of a tetrahedron, it is necessary also to indicate the vertex opposite the face bearing the map. Thus the triangle 123, with the number 4 in its center, refers to the tetrahedron 1234, of which an orthogonal projection has been made onto the face 123. The complete drawing, lined for division of the components into 20 parts, is shown in Fig. 4.

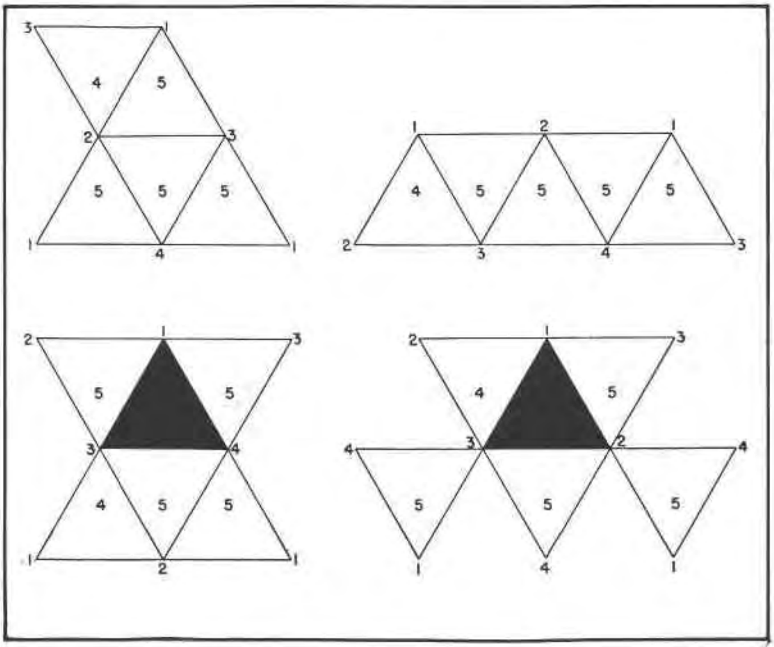

FIG. 3. Four arrangements of the bases of five bounding tetrahedra.

Exactly the same method is used in charting the quartets of 6 variables, but 15 instead of 5 triangles are required, on which to project the contour maps or space curves from the bounding tetrahedra. Only one satisfactory arrangement of these 15 triangles was found. The complete drawing, lined for division of the components into 20 parts, is shown in Fig. 5.

A method of direct projection from hypertetrahedra of four and five dimensions has also been mentioned. Consider first the hypertetrahedron of four dimensions, in which occur sets of points designated as (α, β, γ, δ, ϵ). It is desired to project orthogonally these points into three dimensions, so that the four-dimensional continuum which they define may be shown as surfaces. If one of these coordinates, say ϵ, is to be eliminated by the projection, it will be found that the quadriplanar coordinates of each of the projected points, referred to one of the bounding tetrahedra, will be $(\alpha+\epsilon/4)$, $(\beta+\epsilon/4)$, $(\gamma+\epsilon/4)$, $(\delta+\epsilon/4)$. A number corresponding to ϵ will be associated with each of the projected points, just as some number is associated with every point projected orthogonally from three

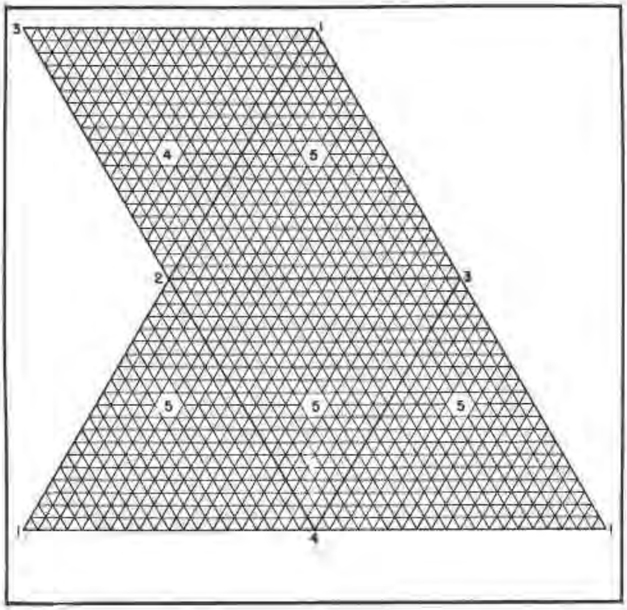

FIG. 4. Bases of five tetrahedra bounding a hypertetrahedron of four dimensions.

into two dimensions. The coordinate eliminated from three dimensions represents altitude, and its various values constitute the data for drawing line contour maps in two dimensions. The coordinate eliminated from four dimensions represents a magnitude of the fourth dimension, and its various values constitute the data for drawing surface contour maps in three dimensions. Therefore in such orthogonal projections, surfaces instead of lines must be passed through or between the projected points. The resulting map is a series of contour surfaces, separated from one another by numerical values corresponding to the eliminated fourth dimension. This surface contour map, by its mode of construction, is automatically referred to one of the tetrahedra that bound the hypertetrahedron of four dimensions.

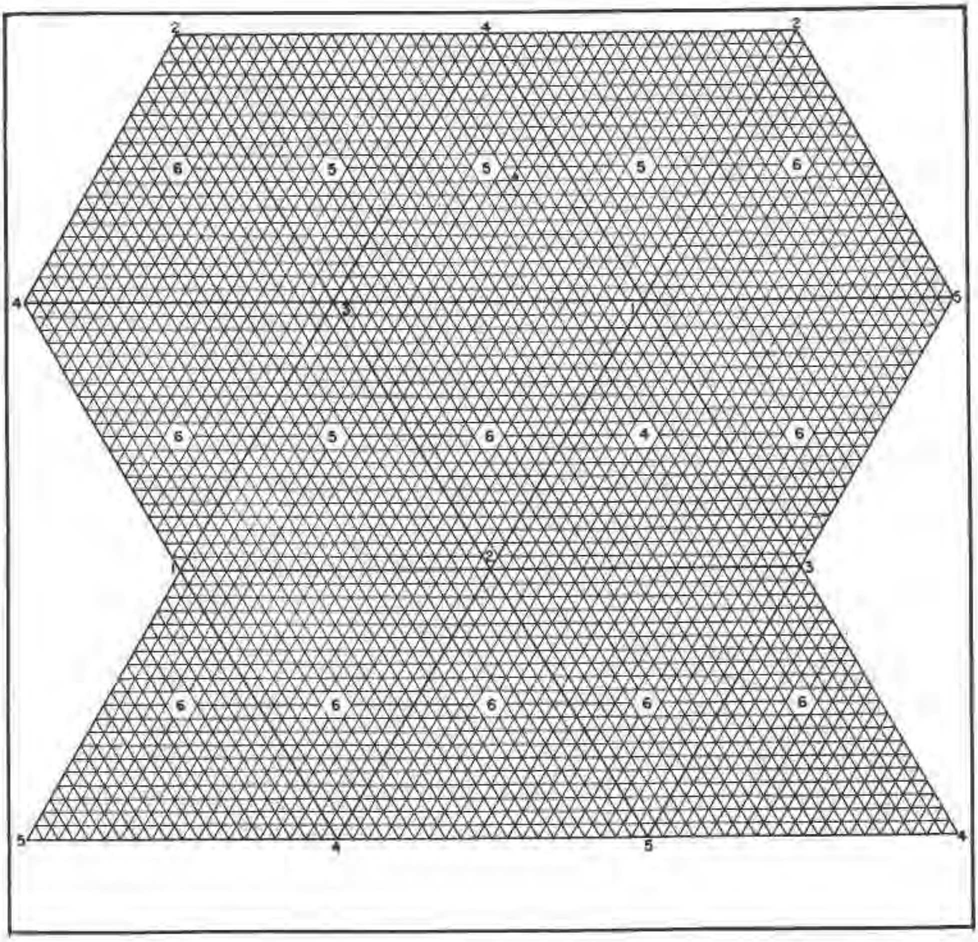

Fig. 5. Bases of fifteen tetrahedra bounding a hypertetrahedron of five dimensions.

This method is difficult of application, not on account of any difficulty in projecting orthogonally from four to three dimensions, but because the subparallel contour surfaces tend to obscure one another. A three-dimensional model must first be constructed, after which several photographs or perspective drawings will be required, in order to see all the contour surfaces and to visualize their relations to one another. One particular condition may exist, however, under which this method would be less troublesome. If the range in the numerical values of the fourth dimension were such that only four contour surfaces would be needed to show advantageously the experimental data, each of these four surfaces could be projected orthogonally onto a face of the tetrahedron of reference. A developed tetrahedron would thus result, showing four line contour maps that would represent the continuum referred originally to the hypertetrahedron of four dimensions.

A continuum defined by points having six coordinates, and designated as $(\alpha, \beta, \gamma, \delta, \epsilon, \xi)$, that is referred to a hypertetrahedron of five dimensions, may theoretically be reduced to surface contours in two different ways. By one method the variables would be recomputed to quintets, and would be charted in multiple with reference to the six bounding hypertetrahedra of four dimensions. As each hypertetrahedron of 4 dimensions is bounded by 5 tetrahedra, this technique would result in the preparation of 30 surface contour maps. A second method would be to make a direct orthogonal projection from five to four dimensions; and to re-project directly from four to three dimensions. The first step would result in a series of four-dimensional continua separated from one another by a magnitude of the fifth dimension; and the second step would result in the production of an exceedingly complex three-dimensional manifold, whereon both the fifth and the fourth dimensions would have to be shown in some manner. The first of these two methods is quite impracticable; the second is too abstruse for serious consideration.

Résumé

Methods are given in this paper for charting five or six variables in relation to the tetrahedra that bound hypertetrahedra of four and five dimensions. A method is also given for projecting directly from a hypertetrahedron of four dimensions into a single bounding tetrahedron; and two methods are outlined for projecting, indirectly and directly, from a hypertetrahedron of five dimensions into hypertetrahedra of four dimensions, and thence into figures of lower dimensions. The most practicable methods for charting five or six variables are by recomputing to triads or quartets, and by plotting these groups in relation to the triangles or tetrahedra that bound the hypertetrahedra.