COMPUTATION OF THE OPTIC AXIAL ANGLE FROM THE THREE PRINCIPAL REFRACTIVE INDICES

FRED. E. WRIGHT, *Geophysical Laboratory, Carnegie Institution of Washington, Washington, D. C.*

ABSTRACT

The standard equation for computing the value of the optic axial angle, $2V_\gamma$, of a biaxial crystal from its three principal refractive indices, α, β, γ, is: $\tan^2 V_\gamma = (1/\alpha^2 - 1/\beta^2)/(1/\beta^2 - 1/\gamma^2)$. This equation is not in convenient form for use in computation. However, an approximate equation, $\cos 2V_\gamma = (\epsilon - \delta)/(\epsilon + \delta) - 6\epsilon\delta/(\epsilon+\delta)(\gamma+\alpha)$, in which $\epsilon = \gamma - \beta$, $\delta = \beta - \alpha$, can be derived from it which is satisfactory and yields values accurate, in general, to 1′ of arc for $2V_\gamma$, if the birefringence, $\gamma - \alpha$, does not exceed 0.050; and to 3′ of arc in case $\gamma - \alpha$ is between 0.050 and 0.100. The simplified formula shows clearly that the optic axial angle is primarily dependent on the difference between the partial birefringences, $\gamma - \beta$, and $\beta - \alpha$, divided by the maximum birefringence, $\gamma - \alpha$, rather than upon the actual values of α, β, γ. The size of the index ellipsoid itself depends upon the values of the principal refractive indices; its shape, on the other hand, depends upon relations between the principal birefringences.

The optical properties of non-opaque biaxial crystals are most readily deduced from the index ellipsoid in which the three principal axes are the refractive indices, α, β, γ, in ascending order of magnitude. In any triaxial ellipsoid there are two diametral plane sections of radius β whose intersections with the ellipsoid are circles. Waves of light propagated along the normals to these sections behave as they would in isotropic substances. These directions are called the optic axes or optic binormals. To find the angle between the two circular sections, note that on the principal α, γ plane of the ellipsoid their traces are the straight lines of radius β. The general equation of the index ellipsoid referred to rectangular axes reads:

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1.$$

For the $\alpha\gamma$ principal section of the ellipsoid, $y = 0$ and the equation becomes

$$\frac{x^2}{\alpha^2} + \frac{z^2}{\gamma^2} = 1. \qquad (1)$$

This equation defines all points of the ellipse on the $\alpha\gamma$ plane of the ellipsoid. For the points at the outer ends of the radius β we have

$$x^2 + z^2 = \beta^2 \quad \text{or} \quad \frac{x^2}{\beta^2} + \frac{z^2}{\beta^2} = 1. \qquad (2)$$

Subtract equation (2) from equation (1) and obtain

$$x^2\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) + z^2\left(\frac{1}{\gamma^2} - \frac{1}{\beta^2}\right) = 0$$

or

$$\frac{z^2}{x^2} = \frac{\frac{1}{\alpha^2} - \frac{1}{\beta^2}}{\frac{1}{\beta^2} - \frac{1}{\gamma^2}} = \tan^2 \Omega.$$

The angle between the normals to the β traces on the $\alpha\gamma$ plane is the optic axial angle $2V_\gamma$ and is the supplement to Ω. Therefore

$$\tan^2 V_\gamma = \frac{\frac{1}{\alpha^2} - \frac{1}{\beta^2}}{\frac{1}{\beta^2} - \frac{1}{\gamma^2}}. \tag{3}$$

This is the standard equation given in text books. Other expressions derived from equation (3) are:

$$\sin^2 V_\gamma = \frac{\frac{1}{\alpha^2} - \frac{1}{\beta^2}}{\frac{1}{\alpha^2} - \frac{1}{\gamma^2}} \quad \text{and} \quad \cos^2 V_\gamma = \frac{\frac{1}{\beta^2} - \frac{1}{\gamma^2}}{\frac{1}{\alpha^2} - \frac{1}{\gamma^2}}. \tag{3a}$$

These three equations are not in a form adapted for computation. Each one expresses an equality between the square of a trigonometric function of half the desired angle and the ratio between differences of the reciprocal squares of the principal refractive indices. A more convenient expression is:

$$\cos 2V_\gamma = \cos^2 V_\gamma - \sin^2 V_\gamma = \frac{\frac{2}{\beta^2} - \left(\frac{1}{\alpha^2} + \frac{1}{\gamma^2}\right)}{\frac{1}{\alpha^2} - \frac{1}{\gamma^2}}. \tag{4}$$

This equation can be simplified with slight loss in accuracy and still yield values of $2V_\gamma$ accurate, in general, to one minute of arc for values of the maximum birefringence $(\gamma - \alpha)$ up to 0.050; and to three minutes of arc for values of $(\gamma - \alpha)$ up to 0.100; and for all values of the refractive index α from 1.400 to 2.000.

Let

$$\beta - \alpha = \delta, \quad \gamma - \beta = \epsilon, \quad \text{and} \quad \gamma - \alpha = \epsilon + \delta; \quad \text{or} \quad \alpha = \beta - \delta, \quad \gamma = \beta + \epsilon.$$

On substituting in equations (3a) and (4) for α and γ the equivalent values $\beta - \delta$ and $\beta + \epsilon$, and neglecting the higher order terms, $2\epsilon^2\delta^2$ and $-4\beta(\epsilon - \delta)\epsilon\delta$, which are, in general, so small that for $(\gamma - \alpha) < 0.050$ they do not influence the result, we obtain the equations:

$$\sin^2 V_c = \frac{\delta}{\epsilon + \delta} + \frac{3\epsilon\delta}{\gamma^2 - \alpha^2} \tag{5a}$$

$$\cos^2 V_c = \frac{\epsilon}{\epsilon + \delta} - \frac{3\epsilon\delta}{\gamma^2 - \alpha^2} \tag{5b}$$

$$\cos 2V_c = \frac{\epsilon - \delta}{\epsilon + \delta} - \frac{6\epsilon\delta}{\gamma^2 - \alpha^2}. \tag{5c}$$

These approximate equations may be used directly if the principal refractive indices are given for a specified wave length of light. Each one consists of two parts, of which the first in (5*c*), $(\epsilon-\delta)/(\epsilon+\delta)$, expresses the ratio of the difference between the two partial principal birefringences to their sum. It should be noted that this ratio alone yields values of the optic axial angle correct, in general, to 1°.5 or less, for refractive index α between 1.400 and 2.000 and for values of the maximum birefringence, $\gamma-\alpha$, from 0 to 0.050.

The ratio $(\epsilon-\delta)/(\epsilon+\delta)$ is simply the difference between the two partial principal birefringences expressed in terms of the maximum birefringence, $\gamma-\alpha$, and is clearly independent of the actual values of the principal refractive indices, α, β, γ. Thus if α_1, β_1, γ_1 refer to one crystal and α_2, β_2, γ_2 to a second, then $\epsilon_1=\gamma_1-\beta_1, \delta_1=\beta_1-\alpha_1$ and $\epsilon_2=\gamma_2-\beta_2$, $\delta_2=\beta_2-\alpha_2$. If now $(\epsilon_1-\delta_1)/(\epsilon_1+\delta_1)$ and $(\epsilon_2-\delta_2)/(\epsilon_2+\delta_2)$ have the same ratio value, $(\epsilon_1-\delta_1)/(\epsilon_1+\delta_1)=(\epsilon_2-\delta_2)/(\epsilon_2+\delta_2)$, then $\epsilon_2/\epsilon_1=\delta_2/\delta_1=C$, wherein C is a constant. For a given value of $\cos 2V_u$, therefore, the actual refractive indices α, β, γ may vary within a wide range, but only in such manner that the proportion $\epsilon_2/\epsilon_1=\delta_2/\delta_1=C$ is maintained and so that the proportionality factor C cancels out in the homogeneous expression. Similar relations obtain for the correction term, $-6\epsilon\delta/(\epsilon+\delta)(\gamma+\alpha)$, and for equations (3*a*) and (4).

If a table of the angle values of $\cos 2V_u$ extending over the range 0 to 1.0 in steps of 0.01 be prepared, the angle corresponding to a given ratio value of $\cos 2V_u=(\epsilon-\delta)/(\epsilon+\delta)$ can be read off directly. The angle $2V_u$ refers to that obtained from the first term of equation (5*c*) alone and without the second term. In table 1, these values are listed in degrees and thousandths of a degree rather than in degrees and minutes of arc. The differences between the angular values for successive steps of 0.01 are also included, so that the actual values of $2V_u$ can be obtained by linear interpolation. For example, let $(\epsilon-\delta)/(\epsilon+\delta)=0.5463$. In table 1 we find $2V_u=57°.316$ for the ratio 0.54; and $2V_u=56°.633$ for the ratio 0.55; the difference between these angles is 0°.683. Therefore the desired value for $(\epsilon-\delta)/(\epsilon+\delta)=0.5463$ is 57°.316 minus $0°.683\times0.63=0°.430$ or 56°.886. Since $1°=60'$, the value in minutes of 0°.886 is $60\times0.886=53'$ and $56°.886=56°53'$. Approximate values of $2V_u$ may be read off directly from figure 1 which is a nomograph of the equation $\cos 2V_u = (\epsilon-\delta)/(\epsilon+\delta)$. The scales for the variables, $(\epsilon+\delta)$, $(\epsilon-\delta)$, and $\cos 2V_u$ are given on the diagram. For example, let $\epsilon+\delta=0.033$; $\epsilon-\delta=0.013$. To find $2V_u$ pass a straight line through the two points and find at its intersection with the $2V_u$ scale, $2V_u=67°$. It should be noted that in this nomogram the cos 2V scale is widely spaced for values near 90°, but very closely spaced near 0°. In other words, the scale is not uniform and more accurate results are obtainable directly from table 1, or from a large scale plot based on this table. This effect of change of scale in

TABLE 1. VALUES OF THE ANGLE $2V_u$ FOR A SERIES OF VALUES OF $\cos 2V_u=(\epsilon-\delta)/(\epsilon+\delta)$ RANGING IN STEPS OF 0.01 FROM 0 TO 1.0

$\frac{\epsilon-\delta}{\epsilon+\delta}$	2 V_u	Diff.	$\frac{\epsilon-\delta}{\epsilon+\delta}$	2 V_u	Diff.	$\frac{\epsilon-\delta}{\epsilon+\delta}$	2 V_u	Diff.	$\frac{\epsilon-\delta}{\epsilon+\delta}$	2 V_u	Diff.
.00	90°000	.573	.25	75°522	.592	.50	60°000	.664	.75	41°410	.874
.01	89°427	.573	.26	74°930	.594	.51	59°336	.668	.76	40°536	.890
.02	88°854	.573	.27	74°336	.596	.52	58°668	.673	.77	39°646	.907
.03	88°281	.573	.28	73°740	.598	.53	57°995	.679	.78	38°739	.925
.04	87°708	.574	.29	73°142	.600	.54	57°316	.683	.79	37°814	.944
.05	87°134	.574	.30	72°542	.601	.55	56°633	.689	.80	36°870	.966
.06	86°560	.574	.31	71°941	.604	.56	55°944	.694	.81	35°904	.989
.07	85°986	.575	.32	71°337	.606	.57	55°250	.701	.82	34°915	1.014
.08	85°411	.575	.33	70°731	.608	.58	54°549	.706	.83	33°901	1.041
.09	84°836	.575	.34	70°123	.610	.59	53°843	.713	.84	32°860	1.072
.10	84°261	.576	.35	69°513	.613	.60	53°130	.720	.85	31°788	1.105
.11	83°685	.577	.36	68°900	.616	.61	52°410	.726	.86	30°683	1.142
.12	83°108	.578	.37	68°280	.618	.62	51°684	.734	.87	29°541	1.183
.13	82°530	.578	.38	67°666	.620	.63	50°950	.742	.88	28°358	1.231
.14	81°952	.579	.39	67°046	.624	.64	50°208	.750	.89	27°127	1.285
.15	81°373	.580	.40	66°422	.627	.65	49°458	.758	.90	25°842	1.347
.16	80°793	.581	.41	65°795	.630	.66	48°700	.767	.91	24°495	1.421
.17	80°212	.582	.42	65°165	.633	.67	47°933	.777	.92	23°074	1.509
.18	79°630	.583	.43	64°532	.636	.68	47°156	.786	.93	21°565	1.617
.19	79°047	.584	.44	63°896	.640	.69	46°370	.797	.94	19°948	1.753
.20	78°463	.585	.45	63°256	.643	.70	45°573	.808	.95	18°195	1.935
.21	77°878	.587	.46	62°613	.647	.71	44°765	.819	.96	16°260	2.190
.22	77°291	.588	.47	61°966	.651	.72	43°946	.832	.97	14°070	2.592
.23	76°703	.590	.48	61°315	.656	.73	43°114	.845	.98	11°478	3.368
.24	76°113	.591	.49	60°659	.659	.74	42°269	.859	.99	8°110	8.110
.25	75°522		.50	60°000		.75	41°410		1.00	0°000	

different parts of a nomogram is common to many nomographs and should be recognized by the user.

In case the refractive indices α, γ are given, and only β is allowed to vary, the maximum birefringence, $\gamma-\alpha$, remains constant, while the partial principal birefringences, $\epsilon=\gamma-\beta$ and $\delta=\beta-\alpha$, vary in opposite directions, such that when $\epsilon=0$, $\gamma=\beta$, the crystal is uniaxial and

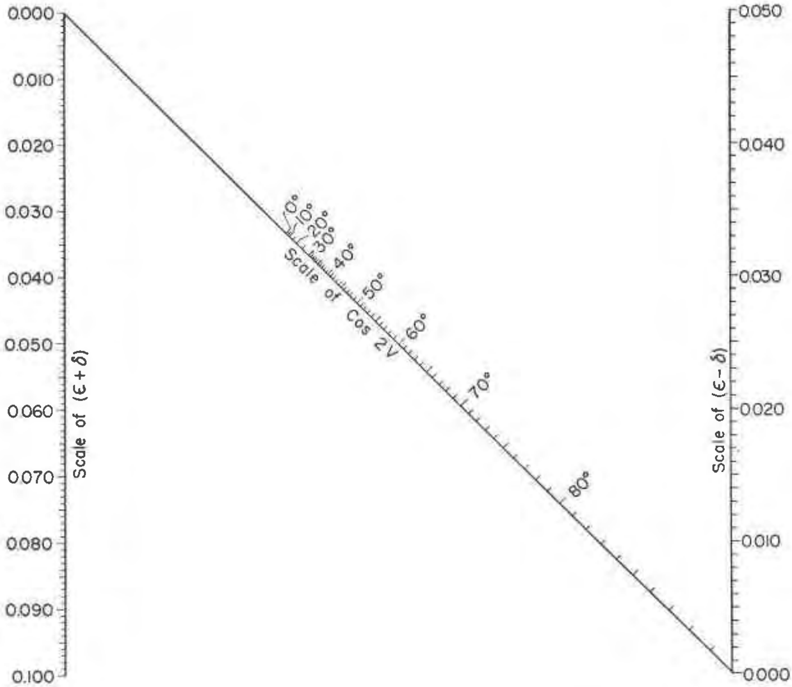

FIG. 1. Nomogram solution of the equation $\cos 2V=(\epsilon-\delta)/(\epsilon+\delta)$. The scale for $(\epsilon-\delta)$ is the ordinate on the right side of the diagram; that for $(\epsilon+\delta)$, on the left side, while that for cos 2V is on the diagonal line of the diagram.

optically negative with α the optical axis; when $\delta=0$, $\beta=\alpha$, the crystal is uniaxial and optically positive with γ the optical axis. In all other cases the crystal is biaxial. When $\epsilon=\delta$, $\epsilon-\delta=0$, the crystal is biaxial and $\cos 2V_u=0$ or $2V_u=90°$.

The second term of equation (5*c*), $-6\epsilon\delta/(\alpha+\gamma)(\epsilon+\delta)$, is of the nature of a correction term (C.T.) by means of which the value of $2V_c$ can in general be determined to 1′ of arc when $(\gamma-\alpha)$ does not exceed 0.050. In case the value of $\cos 2V_c$ is extremely small (0.05 to 0), the error may be greater, because the angle $2V_c$ changes rapidly for slight changes in the value of $\cos 2V_c$ as it approaches zero and linear interpolation in that region is inexact. The correction term itself consists of a constant, $6/(\alpha+\gamma)$, multiplied by the ratio $\epsilon\delta/(\epsilon+\delta)$ in which the numerator is the product $\epsilon\delta$ and the denominator, the sum $(\epsilon+\delta)$. It was proved in a preceding paragraph that for a definite value of $(\epsilon-\delta)/(\epsilon+\delta)$ various partial principal birefringences are possible; but that, to maintain the ratio constant, both ϵ and δ must increase or decrease in the same proportion; and also that $(\epsilon_2+\delta_2)=C(\epsilon_1+\delta_1)$. Therefore, $\epsilon_2\delta_2/(\epsilon_2+\delta_2) = C^2\epsilon_1\delta_1/C(\epsilon_1+\delta_1) = C\epsilon_1\delta_1/(\epsilon_1+\delta_1)$; in other words, the value of the ratio is proportional to the increase of $(\epsilon_2+\delta_2)$ over $(\epsilon_1+\delta_1)$. It follows that if, for a given value of $(\epsilon_1-\delta_1)/(\epsilon_1+\delta_1)$, we plot the change in the value of the fraction $\epsilon_2\delta_2/(\epsilon_2+\delta_2)$ against the change in the maximum principal birefringence, $\epsilon_2+\delta_2=\gamma_2-\alpha_2$, the result will be a straight line passing through the point, $\epsilon_1\delta_1/(\epsilon_1+\delta_1)$. It is possible, therefore, to compute the values of the correction term for one maximum principal birefringence, such as $\gamma_1-\alpha_1=\epsilon_1+\delta_1=0.010$ and from them to obtain the values for any other maximum birefringence and for a given refractive index α_2 by multiplication by the factor $(\gamma_2-\alpha_2)/0.010=C$.

In table 2, the results of the computation are listed for each change of 0.1 in the ratio $(\epsilon-\delta)/(\epsilon+\delta)$; in this table the values listed are given directly in terms of the angle value of $2V_u$ corresponding to the change in the decimal value of $\cos 2V_u$. This is permissible because, for the small changes under consideration, the rates of change in the decimal values of $\cos 2V_u$ are, as a rule, proportional to the changes in the values of $2V_u$ itself.

To compute the value of the correction term for a given refractive index α', and for a given birefringence, $(\gamma'-\alpha')$, when the value of the term is known for another refractive index, α, and for the same birefringence, $(\gamma-\alpha)$, multiply the known value by the factor of proportionality $1/C=\alpha/\alpha'$. For example, let $\alpha=1.400$, $\gamma=1.410$, and $\gamma-\alpha=0.010$; for $(\epsilon-\delta)/(\epsilon+\delta)=0.80$, the value of the correction term is found to be $0.0019217=0^\circ.300$. For the refractive index, $\alpha'=2.000$, we have $\alpha/\alpha' =1.400/2.000=0.7$, and find for the correction term the value, $0.7\times0.0019217=0.0013452=0^\circ.210$. The computed value is that of the exact equation (4) minus the value of the first part of equation (5*c*) and is $0^\circ.212$.

In principle this method is similar to that advocated by E. S. Larsen, Jr. (1921) who prepared a correction chart based on the refractive index 1.500 to serve in finding the correction to be applied to the value of $2V_\gamma$ or $2V_\alpha$ obtained by use of the approximate formula of Mallard (1884, p. 413).

By use of tables 1 and 2, it is thus possible to read off directly the approximate value of $2V_c$ (subscript *c* signifies 'computed') for any values

TABLE 2. ANGLE VALUES OF THE SECOND OR CORRECTION TERM, $-6\epsilon\delta/(\epsilon+\delta)(\gamma+\alpha)$, IN EQUATION (5c) FOR VARIOUS REFRACTIVE INDICES α FROM 1.400 TO 2.000 AND FOR VARIOUS VALUES OF $(\epsilon-\delta)/(\epsilon+\delta)$ IN STEPS OF 0.10 FROM 0 TO 1.0

$\gamma-\alpha=0.010$

$\frac{\epsilon-\delta}{\epsilon+\delta}$	α	1.400		1.450		1.500		1.550		1.600		1.650		1.700		1.750		1.800		1.850		1.900		1.950		2.000	
	$2V_\alpha$	C.T.	D.	C.T.	D.	C.T.	D.	C.T.	D.	C.T.	D.	C.T.	D.	C.T.	D.	C.T.	D.	C.T.	D.	C.T.	D.	C.T.	D.	C.T.	D.	C.T.	D.
			.0		.0		.0		.0		.0		.0		.0		.0		.0		.0		.0		.0		.0
0.0	90°.000	.306°		.295°		.286°		.276°		.268°		.260°		.252°		.245°		.238°		.232°		.226°		.220°		.214°	
			02		01		02		01		02		02		01		02		01		01		02		02		01
0.1	84°.261	.304		.294		.284		.275		.266		.258		.251		.243		.237		.231		.224		.218		.213	
			04		03		04		03		04		04		04		03		04		04		03		03		03
0.2	78°.463	.300		.289		.280		.272		.262		.254		.247		.240		.233		.227		.221		.215		.210	
			08		07		07		08		06		06		06		06		06		06		06		05		05
0.3	72°.542	.292		.282		.273		.264		.256		.248		.241		.234		.227		.221		.215		.210		.205	
			12		12		12		11		11		10		10		10		09		09		09		09		09
0.4	66°.422	.280		.270		.261		.253		.245		.238		.231		.224		.218		.212		.206		.201		.196	
			15		15		14		14		13		13		13		12		13		12		11		11		11
0.5	60°.000	.265		.255		.247		.239		.232		.225		.218		.212		.206		.200		.195		.190		.185	
			21		19		19		18		18		17		17		16		16		15		15		14		14
0.6	53°.130	.244		.236		.228		.221		.214		.208		.201		.196		.190		.185		.180		.176		.171	
			26		25		24		24		23		23		21		21		20		20		19		19		18
0.7	45°.573	.218		.211		.204		.197		.191		.185		.180		.175		.170		.165		.161		.157		.153	
			35		34		33		32		31		30		29		28		27		26		26		25		25
0.8	36°.870	.183		.177		.171		.165		.160		.155		.151		.147		1.43		.139		.135		.132		.128	
			50		49		47		45		44		42		41		41		40		38		37		36		35
0.9	25°.842	.133		.128		.124		.120		.116		.113		.110		.106		.103		.101		.098		.096		.093	
			.1 33		.1 28		.1 24		.1 20		.1 16		.1 13		.1 10		.1 06		.1 03		.1 01		98		96		93
1.0	0°.000	.000		.000		.000		.000		.000		.000		.000		.000		.000		.000		.000		.000		.000	

of the principal refractive indices which are likely to occur. Experience has shown that in certain cases time is saved by disregarding the tables and by using equation (5*c*) directly, either together with a table of logarithms or with a calculating machine and a table of natural trigonometric functions.

In connection with the usefulness of the above method, the relative frequency of the occurrence of maximum principal birefringences, $\gamma-\alpha$, in various minerals and in crystals prepared in the laboratory is important. Thus, of 887 biaxial minerals listed by Larsen and Berman (1934) and of 154 inorganic crystallized salts listed by Fry (1922) the total number for which $(\gamma-\alpha)$ is between 0.001 and 0.010 is 178: between 0.011 and 0.020 it is 245, and so on.

In table 3 the distribution is shown of the numbers of minerals and of inorganic salts which have a given range in principal birefringence, $\gamma-\alpha$; under each heading the percentage distribution is listed also. The last two columns give the total number, minerals plus inorganic salts, and the cumulative percentages. The table shows that more than three fourths of the biaxial crystals (positive and negative) have a maximum birefringence, $\gamma-\alpha$, between 0.001 and 0.050. For roughly 14 per cent of the crystals, the birefringence, $\gamma-\alpha$, is between 0.051 and 0.100; while, for values of $(\gamma-\alpha)$ exceeding 0.100, the percentage is about 11. It follows, therefore, that the use of equations 5*a*, 5*b*, 5*c* is justified in about three-fourths of biaxial minerals and inorganic crystals whose birefringence, $\gamma-\alpha$, ranges between 0.001 and 0.050. The accuracy in the value of 2V is then about 1′ of arc; and for roughly nine-tenths of biaxial crystals for which $(\gamma-\alpha)$ is between 0.001 and 0.100, it may reach 3′ or 4′ of arc.

Equation (5*c*) shows that in case $\epsilon=\delta$ or $(\epsilon-\delta)/(\epsilon+\delta)=0$, $\cos 2V_u=0$ and $2V_u=90°$, but the second term (C.T.) is negative and hence $2V_c$ is greater than 90°; α is then the acute bisectrix and the crystal is optically negative. In general, biaxial crystals for which the partial birefringence, $\beta-\alpha$, is greater than the partial birefringence, $\gamma-\beta$, or $\delta>\epsilon$, are optically negative and α is the acute bisectrix. Biaxial crystals for which $(\gamma-\beta)$ is greater than $(\beta-\alpha)$, or $\epsilon>\delta$, are, as a rule, optically positive with γ the acute bisectrix.

In this connection it is of interest to ascertain how much the midway value of $\beta=(\alpha+\gamma)/2$ may depart from equality to convert the crystal from one of optically negative character to one of positive character. To find the refractive index β for which $\cos 2V=0$ when α and γ are given, we observe from equation (4) that then

$$\frac{2}{\beta^2}=\frac{1}{\alpha^2}+\frac{1}{\gamma^2}. \qquad (6)$$

TABLE 3. RELATIVE FREQUENCY OF BIAXIAL CRYSTALS (MINERALS AND INORGANIC SALTS) FOR VARIOUS PRINCIPAL BIREFRINGENCES ($\gamma-\alpha$) IN INTERVALS OF 0.010 FROM 0 TO 0.100, AND FOR LARGER INTERVALS IN CRYSTALS OF STRONGER BIREFRINGENCE. IN THE LAST TWO COLUMNS OF THIS TABLE THE RELATIVE FREQUENCY IN THE SAME SET OF MINERALS IS LISTED WITH REFERENCE TO THE LEAST PRINCIPAL REFRACTIVE INDEX α

$\gamma-\alpha$	Minerals		Inorganic Salts		Total		α	Minerals	
	No.	Per Cent	No.	Per Cent	No.	Cumulative Per Cent	1. 1.	No.	Cumulative Per Cent
0.001–.010	146	16.5	32	20.8	178	17.1	300–399	7	0.1
.011–.020	201	22.7	44	28.6	245	40.6	400–499	100	11.8
.021–.030	167	18.8	25	16.2	192	59.1	500–599	236	37.9
.031–.040	89	10.0	7	4.5	96	68.3	600–699	281	69.0
.041–.050	63	7.1	12	7.8	75	75.5	700–799	153	86.0
	666	75.1	120	77.9	786			777	
.051–.060	37	4.2	4	2.6	41	79.4	800–899	37	90.0
.061–.070	29	3.3	1	0.6	30	82.3	900–999	20	92.3
.071–.080	26	2.9	5	3.2	31	85.3	2. 2.		
.081–.090	24	2.7	3	1.9	27	87.9	000–099	16	94.0
.091–.100	14	1.6	3	1.9	17	89.5	100–199	19	96.1
	130	14.7	16	10.2	146			92	
.101–.150	47	5.3	8	5.2	55	94.8	200–299	12	97.5
.151–.200	21	2.4	4	2.6	25	97.2	300–399	10	98.6
.201–.250	9	1.0	3	1.9	12	98.4	400–499	5	99.1
.251–.300	4	0.5	1	0.6	5	98.8	500–749	4	99.6
.301–.400	7	0.8	2	1.3	9	99.7	750–999	1	99.7
.401–.500	1	0.1	0	0.0	1	99.8	3. 3.		
.501–.600	1	0.1	0	0.0	1	99.9	000–999	3	100.0
.601–1.20	1	0.1	0	0.0	1	100.0			
	91	10.3	18	11.6	109			35	
Total	887	100.1	154	99.7	1041			904	

For $\alpha=1.500$ and for $\gamma-\alpha=0.010$, 0.020, 0.030, 0.040, 0.050, and 0.100 respectively, we find from equation (6) the values to be: $\beta=1.504975$, 1.509901, 1.514777, 1.519605, 1.524385, and 1.547582. In these cases the differences between the values listed and the midway values of β are, respectively: 0.000025, (0.08′), 0.000099 (0.30′), 0.000223 (0.78′) 0.000395 (1.36′) 0.000615 (2.11′), 0.002418 (8.32′). For $\alpha=2.000$, the corresponding differences are, respectively: 0.000019 (0.07′), 0.000075 (0.26′), 0.000167 (0.58′), 0.000297 (1.02′), 0.000463 (1.59′), 0.001829 (6.29′). These figures prove that with increase in $(\gamma-\alpha)$, the departure of 2V from 90° increases appreciably, but that the total departure does not exceed 10′ of arc. If $\gamma-\alpha=0.100$, the actual change in β may exceed 0.002, a quantity which is easily measurable and which might cause trouble were the above rule on the optical character of a biaxial crystal followed literally. However, the chance of error from this source is slight and is likely to occur only in biaxial crystals of strong birefringence.

The data of tables 1 and 2 together with the linear relations between the birefringence, $\gamma-\alpha$, and the second term of equation (5*c*) can be represented by three simple charts. Experience has proved, however, that the degree of accuracy obtainable by their use is appreciably less than that from equation (5*c*) or from the tables. In many cases the accuracy attainable from the first term alone of equation (5*c*) is quite adequate, especially in crystals of low birefringence and for which the principal refractive indices are given only to the third decimal place. In these cases a shift of only one unit in the third decimal place may produce a change in the value of the optic axial angle of 10° or 20° or even 30°. For this reason it is not surprising in tables of the optical properties of biaxial crystals to note a wide discrepancy between the measured optic axial angle, 2V, listed and that computed from either the exact equation (3) or (4) or from the approximate equation (5) given above. In general and for many purposes, the value $2V_u$ computed from the first term of equation (5*c*) suffices. This term alone shows that the value of the optic axial angle depends chiefly on the difference of the two partial principal birefringences, $(\gamma-\beta)-(\beta-\alpha)$, divided by the maximum principal birefringence $(\gamma-\alpha)$. As a result, the value of 2V is extremely sensitive to slight changes in the values of the principal refractive indices, especially of β. It is, therefore, not advisable to use the optic axial angle in conjunction with any two principal refractive indices to find the third refractive index. If α, γ, and 2V are given, the chances of ascertaining β with fair accuracy are better than when α, β, 2V, or β, γ, 2V are given to find γ or α, respectively.

Historical. Several investigators have sought to simplify the computations involved in equations (3) and (3*a*). Mallard (1884) proposed that in equations (3) and (3*a*) the principal birefringences, $\gamma-\alpha$, $\gamma-\beta$, $\beta-\alpha$, be substituted for the differences between the reciprocal squares of the corresponding principal refractive indices. He noted that for biaxial crystals of medium to weak birefringence the approximation is in general sufficiently close to the exact value to be satisfactory. In line with Mallard's suggestion the expression in equation (4) may be changed to read: $\cos 2V_\gamma = \cos^2 V_\gamma - \sin^2 V_\gamma = (\gamma-\beta)/(\gamma-\alpha) - (\beta-\alpha)/(\gamma-\alpha) = (\gamma+\alpha-2\beta)/(\gamma-\alpha)$. Computations show that this equation for $2V_u$ yields values for 2V correct to 1°5 for crystals whose birefringence, $\gamma-\alpha$, does not exceed 0.050. With increase in birefringence, $\gamma-\alpha$, the degree of accuracy decreases appreciably; thus for $\alpha=1.500$ and $\gamma-\alpha=0.100$ the error is 2°773; for $\alpha=2.000$ and $\gamma-\alpha=0.100$, it is 2°096.

In 1911 (Plate 9) Wright published a graphical chart based on the equation $\sin^2 V_\gamma = (\beta-\alpha)/(\gamma-\alpha)$ from which, having given $\beta-\alpha$ and $\gamma-\alpha$, the value of V_u can be read off directly. The chart extends to values of $\gamma-\alpha=0.090$ and $\gamma'-\alpha'=0.090$. It would have been better had the chart been extended to $\gamma-\alpha=0.100$ and $\gamma'-\alpha'=0.100$, and in the labeling of the optic axial angle, had $2V_u$ been used for V_u.

In 1912 Boldyrew published three diagrams based on the exact equation $\tan V_\gamma = \gamma\sqrt{\beta^2-\alpha^2}/\alpha\sqrt{\gamma^2-\beta^2} = (\beta+\epsilon)\sqrt{\beta^2-(\beta-\delta)^2}/(\beta-\delta)\sqrt{(\beta+\epsilon)^2-\beta^2}$. When β is known, this equation defines relations between the partial birefringences $\epsilon=\gamma-\beta$, $\delta=\beta-\alpha$, and V_γ. If V_γ be known, then for any value of ϵ a corresponding value of δ is given. Let the values of ϵ be the abscissae and those of δ the ordinates. A series of curves for V_γ is thus obtained by computation which enables the observer to read off directly the value of the third variable when the other two are known. For each diagram one β is valid, namely, $\beta=1.500$, $\beta=1.650$, and $\beta=2.000$. These charts are interesting, but they have not come into general use.

In 1913 (Plates VI and VII) F. E. Wright published two charts for the solution of the equation (3), (Plates VI and VII); with these a table of the values of reciprocal squares of refractive indices was included. In Plate VI the values of $1/\alpha^2-1/\beta^2$ are abscissae, those of $1/\beta^2-1/\gamma^2$, the ordinates, and the series of straight lines radiating from zero, the values of V_α. In Plate VII the values of $(1/\alpha^2-1/\beta^2)^{1/2}$ are the abscissae, those of $(1/\beta^2-1/\gamma^2)^{1/2}$ the ordinates, while the radiating straight lines denote the values of V_α. In Plate VII the distribution of the V_α values is more uniformly spread and for this reason is superior to Plate VI. It should be noted that both Plates serve equally well for the approximate equations $\tan^2 V\alpha = (\gamma-\beta)/(\beta-\alpha)$ and $\tan V\alpha = [(\gamma-\beta)/(\beta-\alpha)]^{1/2}$.

In 1927 Roesch and Sturenburg suggested a modification of the exact equation (3) and expressed it in the form $\tan^2 V_\gamma = (\gamma/\alpha)^2(\beta/\alpha)^2-1)/(\gamma^2/\alpha^2-\beta^2/\alpha^2)$ in which the ratios, β/α and γ/α, are given. In order that this form be useful, the writers computed a series of tables for the two ratios and presented the solutions of the equation in two graphs of curves from which the value of the optic axial angle can be read off directly. This method, though interesting, has not been used greatly.

In 1937 Smith, on the basis of Plate VI of the above paper by F. E. Wright and for the approximate equation, $\tan^2 V_\gamma = (\beta-\alpha)/(\gamma-\beta)$, published a new diagram by which, with the aid of a sliding scale, he was able to read off directly and without computation the values of the partial birefringences or of their ratio. In his chart a central vertical line divides it into a positive and a negative section. On the sliding scale, which is divided uniformly into convenient units, the value of α on the scale is placed on the left of the central line; the scale division for β is placed at the central mark on the base line of the diagram, and that of γ on the right of β. The intersection of the N.E. diagonal line through α on the scale with the N.W. diagonal through γ determines the position of the straight line β and thereby the angle V_γ (labeled $2V_\gamma$ for convenience), and also the optical character of the crystal, whether positive or negative. Smith's chart is convenient, but, as Smith emphasizes, its accuracy is not high for weak birefringences because of the clustering of the lines near their point of intersection.

In 1938 Lane and Smith described another chart for the solution of the approximate equation. This chart requires the use of a sliding scale but of a different kind. The chart is said to be more satisfactory for crystals of low birefringence than is the Smith chart. However, neither chart has been used widely.

In 1942 Mertie described a nomographic chart based on the exact sine equation (*3a*). The Mertie chart is self-contained in the sense that if the three principal refractive indices are known, the optic axial angle and the optical character can be read off directly by use of a straight edge or of a straight line and without computation of any kind. In the nomograph the horizontal scale, which determines the spacing of the vertical lines, is $\sin^2 V_\gamma$; the vertical scale, which determines the spacing of the horizontal lines, is $1/n^2$. On the left side of the plot the scale is $1/\alpha^2$ or $1/\beta^2$, but it is marked α or β; on the right side the scale is the same as that on the left (function $\sin^2 V_\gamma$), but it refers to $1/\gamma^2$ or $1/\beta^2$ and is labeled γ or β. Similarly V_γ extends on the left side of the diagram from 0° to 45° and is marked positive; on the right side of the diagram, V_α extends in the reverse direction from 0° to 45° and is marked negative. Since the user of the diagram is interested chiefly in $2V_\gamma$ and $2V_\alpha$, it might have been wise to label on the positive side the angles $2V_\gamma$ as from 0° to 90°, and similarly on the negative side.

In a second nomograph Mertie solves directly the equation $\sin^2 E/\beta^2 = (1/\alpha^2 - 1/\beta^2)/(1/\alpha^2 - 1/\gamma^2)$, thus avoiding the need for a second nomograph for the equation $\sin E = \beta \sin V$.

It is obvious that because of the closer spacing of the horizontal lines with increase in value of α and because the scale for $\sin^2 V_\gamma$ is more closely crowded for small values of V_γ, the degree of accuracy of the plot varies appreciably in its several sections. For minerals of low birefringence, say $\gamma-\alpha=0.010$, the error in determination of $2V\gamma$ may be several degrees because of the acute angle between a horizontal line and the straight line passed through α on the left side and through γ on the right side of the diagram. Under these conditions a very small error in the spacing of the horizontal lines on the chart produces a significant error in the optic axial angle. As the result of a number of test readings on the chart and a comparison with the values obtained by direct computation by the exact formula, it may be stated that the error made by use of the Mertie chart is in general somewhat less than that obtained by use of the simple chart of figure 1, based on the approximate Mallard equation. The average error of the readings of 2V from the Mertie chart for various refractive indices and various birefringences was one degree. Many of the readings were too high; some were too low, as might be expected.

In 1945 Waldmann published a nomogram based on equation (3). By reducing the index ellipsoid to one in which the β-axis is unity and adopting the ratios, $A=\alpha/\beta$ and $C=\gamma/\beta$, whereby α and γ are expressed in terms of β, he obtains from equation (3) the form $\tan^2 V=(1/A^2-1)/(1-1/C^2)$, in which two independent variables instead of three occur. To find the values of $1/A^2$ and $1/C^2$ he plots the values of β^2 along the ordinate axis and $1/A^2$ and $1/C^2$ along the abscissa, with the origin of coordinates on the right. Through the ordinate, $n^2=0$ at abscissa $x=0$, radiating lines are drawn across the diagram. The intersections of these lines with the horizontal line through the ordinate at β^2 yield the values of $(1/A^2-1)$ on the left of ordinate at $x=1$ and of $(1-1/C^2)$ on the right. These values are labeled A and C; for convenience, the scale of the x-axis is made 4-times that of the y-axis. To find the optic axial angle a second diagram is used which consists of a right angle triangle whose sides include an angle of $+45°$ and $-45°$ respectively with the ordinate at $x=1$. The sides of the triangle are graduated in units of the abscissae of the first diagram increased in the amount, $1/\cos 45°$.

Theoretically the nomogram is correct and interesting; but for small values of V it is unsatisfactory in practice, even when only a portion of the original diagram is used and a correspondingly larger scale is adopted, as has been done by Burri (1950, p. 49 and Plate 1). The crowding of the diagonal lines through the apex of the triangle might be avoided by use of a nomogram based on equation (4); but experience has proved that the Mertie nomogram is better suited to the purpose.

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