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REFRACTION CORRECTIONS FOR UNIVERSAL STAGE MEASUREMENTS. I. UNIAXIAL CRYSTALS

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Abstract

Snell's law is not generally valid for correcting crystallographic orientations measured optically with the universal stage in cases where refractive indices of crystal and hemispheres are unequal. Analysis of the refraction correction for uniaxial crystals shows that when the optic axis is oriented ostensibly perpendicular to the microscope stage, the Snell's law correction is valid to a good approximation, but it fails drastically when the optic axis is set ostensibly parallel to the microscope stage. In the latter case the true correction to the measured orientation can even be in the opposite direction to that given by Snell's law. The exact correction formula for this case is complicated, but a linear approximation proves to fit the exact correction curve rather well over the entire measurement range. The linear formula shows that in most practical applications where hemispheres are used, the true correction to measured orientations in which the optic axis is set parallel to the microscope stage is negligibly small, even when the Snell's law correction would be large. In special cases, particularly ice petrofabric measurement, the corrections are not negligible. The theoretical correction curves are tested experimentally by measurements using quartz and ice, and are well substantiated. The theoretical method can be extended to biaxial crystals, but that is not done here. The availability of correction curves valid for large refractions makes it possible to dispense with hemispheres, which can be advantageous in petrofabric work involving large numbers of thin sections.

Introduction

It has always been taken for granted that Snell's law is to be used in correcting crystal orientations measured with the universal stage, when indices of refraction of crystal and hemispheres do not coincide. In the present paper it is shown that this assumption is not generally valid and that it leads in many cases to large errors. Theoretical refractive-index correction curves, differing in part greatly from the Snell's law curve, are derived for uniaxial crystals and are verified experimentally. Extension of the treatment to biaxial crystals is possible but is not carried out here, although a few remarks bearing on this extension are given.

Discovery of the failure of the Snell's law orientation-correction is the result of petrofabric study of glacier ice. In such work, hemispheres cannot be used on the large universal stage that is required, and the index-of-refraction correction of the measured orientations is therefore particularly large. Rigsby (1951, p. 597) observed that the orientations of ice crystals whose c-axes are inclined at about 45° to the plane of the thin section can often be measured both by setting the c-axis parallel to the line of sight ("vertical") and perpendicular to the line of sight ("hori-

[1] Contribution No. 1006.

zontal"); when the two such measurements for a given crystal are corrected by Snell's law and the resulting orientations compared, they always disagree by about 10° and in such a way as to indicate that the correction applied to one or both of the measurements is too large. I noticed the same thing later in my ice petrofabric work using Rigsby's equipment, but at first I put the discrepancy down to the inherent difficulties and uncertainties in measuring high tilt angles with the relatively crude technique involved. However, Rigsby carried out in the laboratory a further investigation of the matter, and found that the Snell's law correction for ice seems to be applicable to the "vertical" settings but not to the "horizontal" ones; the results of this work were mentioned by Langway (1958), but no detailed report was published. Although it had been suggested (Rigsby, 1951, p. 598) that biaxial character or other anomalous optical properties of the ice crystals studied might be responsible for the peculiar effects, it turns out, as shown in the present paper, that a general phenomenon is involved, which will be encountered in all uniaxial and also in biaxial crystals.

That the tilt angles measured by setting the *c*-axis "vertical" require a different correction-law from those measured by setting *c* "horizontal" seems at first sight to have a ready explanation in the fact that the "vertical" settings can be made, and therefore ordinarily are made, with the help of only a slight tilt on the *EW* (A_4) universal stage axis, whereas the "horizontal" settings require a relatively large tilt on this axis. Examination of this idea requires an analysis of the optical principles by which crystal orientations are measured with the universal stage. This analysis, given in the following sections, leads unexpectedly to a refutation of the foregoing explanation and shows that the effect is a general one, defying ordinary intuition. The tilt on the *EW* axis plays only a minor role.

Principles of Crystal Orientation Measurement with the Universal Stage

All methods of measuring optical orientation with the universal stage depend on the phenomenon of extinction. In the orthoscopic methods (Emmons, 1943, pp. 23–39), manipulation of the stage axes is controlled by setting the crystal under observation to extinction (minimum transmitted light), which corresponds in the conoscopic methods (Hallimond, 1950) to placing an isogyre at the center of the conoscopic field. There is thus a close relationship between the conditions that relate universal stage manipulation to crystal orientation and the conditions that determine the location of isogyres in interference figures. The latter have been investigated by Kamb (1958), where it is shown that minimum transmitted light at a given point in the interference field is achieved when, for the corresponding direction of light-propagation in the crystal plate under observation, the vibration directions, stereographically projected (as described below) onto the plane of the microscope stage, coincide with the vibration directions transmitted by polarizer and analyzer. This condition for minimum-light is the correct one when the rotation of the polarization plane of the incoming light due to refraction at interfaces below the crystal plate is equal to the further rotation in the same sense due to refraction above the crystal plate. The latter requirement is well fulfilled under universal stage observation, because, if the mounting oil is matched to the index of refraction of the glass platform of the universal stage, and if the indices of refraction of cover glass and of the glass slide on which the thin section is mounted are equal, then the sequence of refractions below and above the crystal plate are just the same, in reverse order. The argument here, which is not given in detail, is a direct consequence of the discussion in the paper cited (Kamb, 1958, pp. 1035–1043), where in equation (9) we have $\delta_1 = \delta_2$, because of (10), and consequently $\xi = 0$.

The foregoing considerations lead to the conclusion that the condition for extinction of a crystal plate under universal stage manipulation is the following. We consider a light ray travelling parallel to the microscope axis, and we consider the corresponding refracted ray in the crystal plate, this refracted ray being in general inclined to the microscope axis if the crystal plate and hemispheres have different refractive indices. For this inclined, refracted ray we imagine a small cross to be constructed whose arms are parallel to the vibration directions for this ray in the crystal plate. We imagine this cross to be placed on the surface of a sphere at the endpoint of a radius vector in the direction of the refracted wave normal, and we assume that the birefringence of the crystal plate is small, so that ray and wave-normal directions essentially coincide. We then stereographically project the cross from the surface of the sphere onto the plane of the microscope stage. When the stereoscopically-projected cross coincides in orientation with the *NS-EW* cross formed by the vibration directions of polarizer and analyzer, the crystal plate is at extinction. At the same time, under conoscopic observation an isogyre passes through the center of the interference field.

This principle will be used in the following sections, in which the refractive index correction for universal-stage measurement of the optic-axis orientation of uniaxial crystals is analyzed. We consider separately the cases when (1) the optic axis is set ostensibly perpendicular to the microscope axis, here called the *H setting*, and (2) the optic axis is set ostensibly parallel to the microscope axis, here called the *V setting*.

Uniaxial Crystals in *H* Setting

In the standard universal stage procedure for measuring the *c*-axis orientation of a uniaxial crystal (Emmons, 1943, p. 23), the situation to be analyzed is the result of the following sequence of operations. (1) Starting with all axes at their null settings the crystal is first rotated about the *IV* stage axis[1] until the *c* axis lies in the "vertical" east-west plane (east-west plane parallel to the microscope axis). (2) This plane in which the *c* axis lies is tilted through some angle, here designated γ, by rotating on the *EW* axis. (3) The *NS* axis (which no longer is parallel to the microscope stage) is now tilted through some angle ζ, until the crystal reaches extinction.

The situation at the completion of this sequence of operations is shown in Fig. 1, which is a stereographic projection whose pole *i* is the pole of the microscope stage, *i.e.*, the direction of the microscope axis. *P* is the pole of the inner stage platform of the universal stage. Initially, before the above sequence of operations, *P* lies at *i*, and in the first operation it remains at *i* while the *c*-axis, whose orientation is designated by *c* in Fig. 1, is brought to lie in the *WPE* plane. This plane is then tilted through the angle γ by rotating on the *EW* axis. Finally, by rotation on the *NS* axis (which remains perpendicular to the tilted *WPE* plane), *P* is moved along circle *WPE* through the angle ζ away from the *NiS* plane. In this last motion *c* and *P* move synchronously along great circle *EPW*, and the position that *c* reaches upon completion of the operation is indicated by the angle ν in Fig. 1.

Now if the final setting upon completion of the above operations is *H* and if the effects of refraction are ignored, then we expect the *c* axis to end up at *W* in Fig. 1, so that $\nu=0$. In this case the inclination of *c* to the plane of the thin section or inner stage platform is just the measured tilt ζ on *NS*. Because of refraction, however, ν will not in general be zero, and inspection of Fig. 1 shows that ν as there defined is the angular correction that must be *subtracted* from the measured tilt ζ to get the true inclination $\zeta-\nu$ of the *c*-axis to the plane of the thin section.

The point where *c* actually ends up, taking refraction into account, is determined in the following way. The incident light ray at *i*, travelling up the microscope axis, is refracted in the crystal plate through an angle θ to *r*, which is the direction of the refracted ray (or, strictly speaking, the refracted wave normal, which under the assumption of small crys-

[1] The Emmons notation for the universal stage axes is used here, the "*O.E.W.*" axis (which is the only *EW* axis of the four-axis stage) being here referred to simply as *EW*. The corresponding Berek notation ($IV=A_1$, $NS=A_2$, $EW=A_4$), while it has some advantages, is more of a burden on the memory.

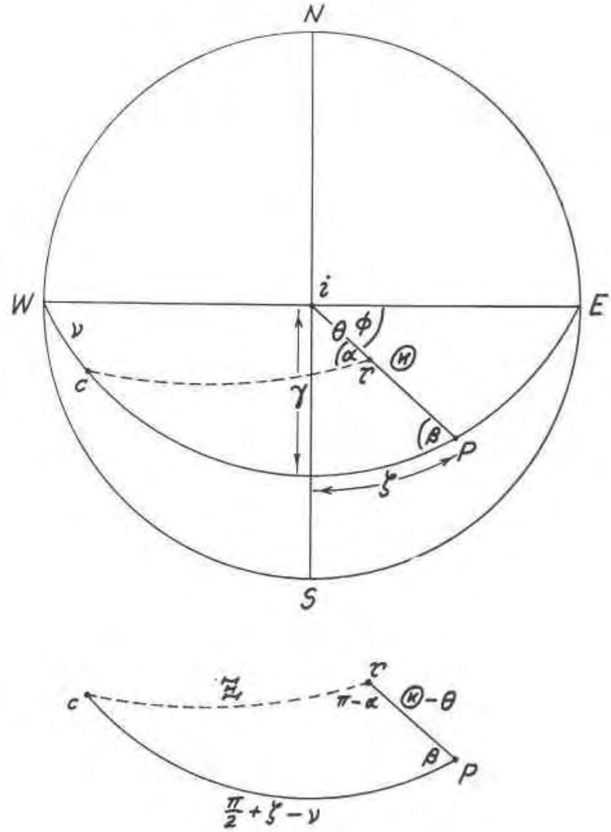

FIG. 1. Stereographic projection showing the geometrical relationships during universal-stage measurement of a uniaxial crystal in H setting. Plane of projection is the microscope stage. Incident light ray from polarizer, vibrating parallel to NS, travels along the microscope axis i and is refracted on entering the crystal plate to r. The pole of the crystal plate (pole of inner stage platform of the universal stage) is at P. γ is the angle through which the stage has been tilted on its EW axis, and ζ the tilt angle on its NS axis. c is the orientation of the optic axis of the crystal, and ν is the correction that must be *subtracted* from the measured tilt ζ to get the true inclination $\zeta-\nu$ of the optic axis to the plane of the thin section. The symbol θ designates the angle represented by arc ir in the projection, and Θ is the angle represented by arc irP. In the lower diagram the spherical triangle crP is reproduced with its sides and vertex angles labelled.

tal birefringence essentially coincides with the ray). r lies in the iP plane a distance θ from i, and the angle Θ denotes the total angular tilt of P away from i, which is determined by the combined tilts γ on EW and ζ on NS. The vibration directions for the refracted ray are determined by the great circle cr. When the tangent to this great circle at r, as seen in stereographic projection, is parallel to the EW axis, then, from the argument of the previous section, the crystal is at extinction; for a given tilt ζ this condition determines where c must lie and hence what value the angle ν must have.

We now proceed to write down the above conditions analytically. There are two physical conditions to be applied—the refraction of the ray and the condition for extinction—and a number of geometrical ones. The analytical form of these conditions will consist of a group of equations that can be solved for the correction angle ν as a function of the tilt angles γ (on EW) and ζ (on NS) and of the indices of refraction of crystal and hemispheres.

The refraction of the wave normal (which under the assumption stated is essentially coincident with the ray) is governed by Snell's law:

$$m \sin (\Theta - \theta) = \sin \Theta \tag{1}$$

where

$$m = \frac{n_{xl}}{n_s} \tag{2}$$

n_s being the refractive index of the hemispherical segments and n_{xl} the mean refractive index of the crystal, the birefringence being assumed small.

The condition for extinction, as given above, is simply

$$\alpha = \phi \tag{3}$$

where these two angles are defined in Fig. 1.

The geometrical conditions are required to relate the various angles involved. From the right spherical triangle involving the sides of lengths γ, ζ, and Θ we have

$$\cos \Theta = \cos \gamma \cos \zeta \tag{4}$$

and

$$\cos\left(\frac{\pi}{2} - \phi\right) = \tan \gamma \cot \Theta \tag{5}$$

and also

$$\cos \beta = \tan \zeta \cot \Theta \tag{6}$$

where the vertex angle β is as defined in Fig. 1. From the large spherical triangle crP, which is reproduced separately in Fig. 1, we have

$$\cos \Xi = \cos\left(\frac{\pi}{2} + \zeta - \nu\right) \cos (\Theta - \theta) + \sin\left(\frac{\pi}{2} + \zeta - \nu\right) \sin (\Theta - \theta) \cos \beta \tag{7}$$

and

$$\frac{\sin \Xi}{\sin \beta} = \frac{\sin\left(\frac{\pi}{2} + \zeta - \nu\right)}{\sin (\pi - \alpha)} \tag{8}$$

where the side length Ξ is as defined in the figure.

The above equations suffice to determine the desired solution. We combine them in the following way. Into the relation

$$\sin^2 \Xi + \cos^2 \Xi = 1 \tag{9}$$

we introduce the values of $\sin \Xi$ and $\cos \Xi$ from (7) and (8), and then with the help of the remaining equations eliminate the quantities α, β, ϕ, θ, and Θ in terms of ζ, γ, and m. After some reduction we arrive at the result

$$\left[\frac{1}{m} \cos(\zeta - \nu) \sin \zeta \cos \gamma - \sin(\zeta - \nu) \sqrt{1 - \frac{1}{m^2} + \frac{1}{m^2} \cos^2 \gamma \cos^2 \zeta}\right]^2 + \frac{\cos^2(\zeta - \nu)}{\cos^2 \zeta} = 1 \tag{10}$$

Letting $x = \sin^2(\zeta - \nu)$ and expanding the bracket, (10) can be written in the form

$$-\sqrt{A}\sqrt{x(1 - x)} = Bx + C \tag{11}$$

where

$$\begin{aligned} A &= \frac{4}{m^2} \sin^2 \zeta \cos^2 \gamma \left(1 - \frac{1}{m^2} \sin^2 \zeta - \frac{1}{m^2} \sin^2 \gamma \cos^2 \zeta\right) \\ B &= \tan^2 \zeta + \frac{1}{m^2} \sin^2 \zeta (1 + \cos^2 \gamma) + \frac{1}{m^2} \sin^2 \gamma \cos^2 \zeta \\ C &= \tan^2 \zeta + \frac{1}{m^2} \sin^2 \zeta \cos^2 \gamma \end{aligned} \tag{12}$$

The solution to (11), found upon squaring, is

$$x = \frac{2BC + A \pm \sqrt{A(A + 4CB - 4C^2)}}{2(B^2 + A)} \tag{13}$$

We now must choose the correct root from among the four given by

$$\sin(\zeta - \nu) = \pm \sqrt{x} \tag{14}$$

where x is given by (13). The extra roots were introduced by squaring (11) and by squaring $\cos \Xi$ and $\sin \Xi$ in (9). We cannot choose the sign in (13) by substituting (13) back into (11), because the extra roots added by squaring (11) correspond to the two possible signs in (14) rather than to the two signs in (13). It would doubtless be possible to resolve the four roots by returning to the original equations (1) to (8), but a more straightforward procedure is available. We know that if $m = 1$ there is no refraction of the light on entering the crystal plate and therefore no correction to the measured orientation ζ is required, so that $\nu = 0$. If we put $m = 1$ in equations (12) and substitute into (13), we arrive, after considerable reduction, at

$$\sin^2(\zeta - \nu) = \sin^2\zeta \frac{1 + \cos^4\gamma\cos^4\zeta + 2\cos^2\gamma\sin^2\zeta\cos^2\zeta \pm 2\cos^2\gamma\cos^4\zeta}{1 + \cos^4\gamma\cos^4\zeta + 2\cos^2\gamma\sin^2\zeta\cos^2\zeta - 2\cos^2\gamma\cos^4\zeta} \tag{15}$$

In order that (15) yield identically $\nu=0$ it is seen that we must take the $-$ sign in (13) and the $+$ sign in (14).

There are now two ways of evaluating the rather cumbersome result in (13). One is straightforward calculation of correction curves $(\zeta-\nu)$ *vs.* ζ for particular values of m and γ; the other is to try, by making suitable approximations, to reduce (13) to an approximate formula that is simpler and more comprehensible.

For purposes of calculation it is useful to bring (13) into a somewhat more convenient form, although without approximation it cannot be reduced to anything really simple. If we define three quantities

$$\omega = \frac{1}{m^2}\sin^2\zeta\cos^2\gamma \tag{16}$$

$$\sigma = \frac{1}{m^2}\sin^2\gamma \tag{17}$$

and

$$\lambda = \tan^2\zeta \tag{18}$$

and also define two functions $f^+(\omega, \sigma, \lambda)$ and $f^-(\omega, \sigma, \lambda)$ by

$$f^\pm(\omega, \sigma, \lambda) = \frac{\lambda^2 + \lambda(3\omega + \sigma) + \omega(2 - \sigma) \pm 2\sqrt{\omega(1 - \omega - \sigma)(\lambda\omega + \lambda\sigma + \omega)}}{\lambda^2 + 4\omega(\omega + 1) + \sigma^2 + 2\lambda(2\omega + \sigma)} \tag{19}$$

then the result in (13) can be written

$$\sin(\zeta - \nu) = +\sqrt{f^-(\omega, \sigma, \lambda)} \tag{20}$$

Curves of $\zeta-\nu$ as a function of ζ, calculated from (16)–(20) for $m=1.548$ and for $\gamma=0°$, 40°, and 60°, are shown in Fig. 2, under the designation *H*. Also shown is the Snell's law correction curve

$$m\sin(\zeta - \nu) = \sin\zeta$$

for the same index-ratio m. It is seen that the theoretical curves given by (16)–(20) differ greatly from the Snell's law curve. In fact the measured tilt ζ is *smaller* than the actual inclination $\zeta-\nu$ of the optic axis to the plane of the thin section, so that the correction ν is *negative*, in the opposite direction to the correction given by Snell's law. From the curves it is seen that the effect of the *EW* axis setting γ is relatively slight, appreciable only at angles γ about 30° and greater. The largest departure of the curve for $\gamma=20°$ from that for $\gamma=0°$ is only 0.4°. Thus the violation of the Snell's law correction remains prominent as $\gamma\rightarrow0$.

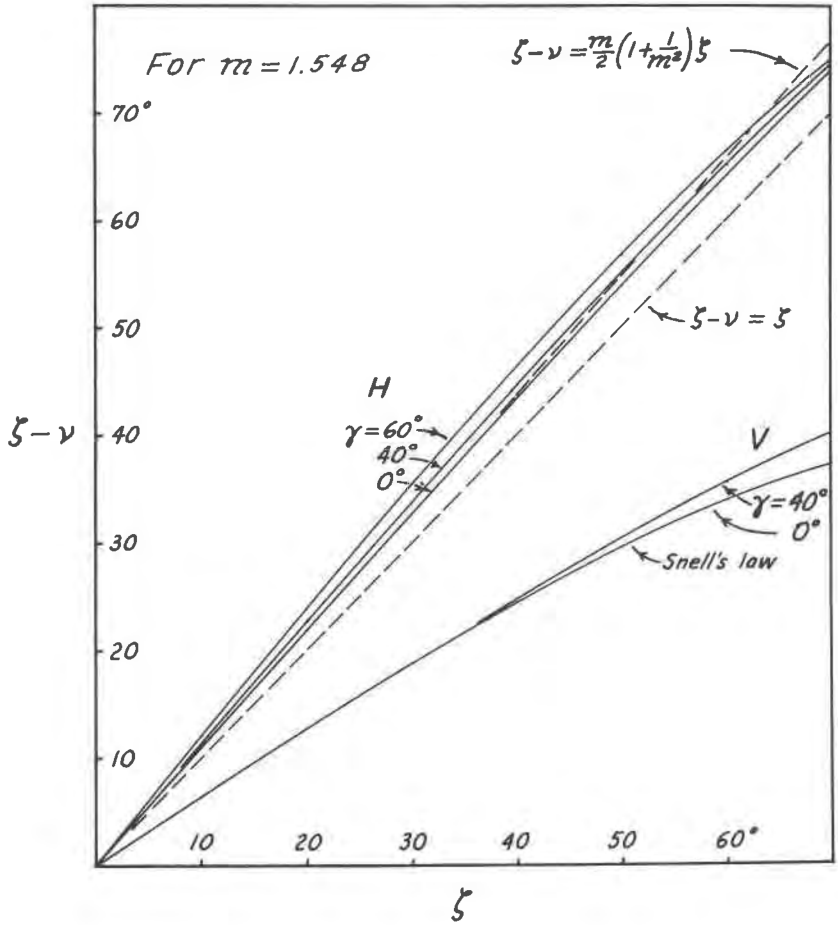

FIG. 2. Theoretical refraction-correction curves for universal stage measurement of crystal orientation, calculated for $m=1.548$. The true (refraction-corrected) angle $\zeta-\nu$ is plotted against the measured angle ζ. The curves for the H setting (equation (20), upper curves), which are given for three values of the tilt angle γ on the EW axis, lie conspicuously above the curve of no correction ($\zeta-\nu=\zeta$). Also shown for comparison (dashed line) is the linear approximation (22). For the V setting (lower curves) the ordinate should read $\zeta-\mu$; the curves for this case are calculated from (31) or, for $\gamma=0$, from (33).

A useful approximate relation between $\zeta-\nu$ and ζ can be obtained by expanding (19) in powers of ζ. In doing this we must distinguish the two cases $\gamma=0$ and $\gamma\neq 0$, because the reduction of (19) in the two cases is very different: in the first, ω dominates in the numerator and denominator, since $\sigma=0$, whereas in the second, σ dominates if we assume $\zeta \ll \gamma$. For $\gamma=0$ the numerator and denominator in (19) are

$$\begin{aligned} Num. &= \zeta^4\left(1+\frac{1}{m^2}\right)^2 + O(\zeta^6) \\ Denom. &= \frac{4}{m^2}\zeta^2 + O(\zeta^4) \end{aligned} \tag{21}$$

so that for small ζ and $\zeta-\nu$ we therefore have

$$\zeta - \nu = \zeta\,\frac{m}{2}\left(1+\frac{1}{m^2}\right) \tag{22}$$

For $\gamma \neq 0$, the lowest-order terms in the expansion of numerator and denominator of (19) in powers of ζ are, letting $\Gamma = \sin\gamma$,

$$\begin{aligned} Num. &= \frac{\zeta^2}{m^2}\left\{2 - \Gamma^2\left(1+\frac{1}{m^2}\right) + \frac{\Gamma^4}{m^2} - 2\sqrt{(1-\Gamma^2)\left(1-\frac{\Gamma^2}{m^2}\right)}\right\} \\ Denom. &= \frac{\Gamma^4}{m^4} \end{aligned} \tag{23}$$

If the square-root in (23) is expanded in powers of Γ we find that the Γ^2 terms drop out and

$$Num. = \frac{\zeta^2}{m^2}\left[\frac{1}{4}\left(1+\frac{1}{m^2}\right)^2\Gamma^4 + \frac{1}{16}\left(1-\frac{1}{m^2}\right)\left(1-\frac{1}{m^4}\right)\Gamma^6 + \cdots\right] \tag{24}$$

To the lowest order in ζ and $\zeta-\nu$, (19) thus reduces for $\gamma \neq 0$ to

$$\zeta - \nu = \zeta\left\{\frac{m}{2}\left(1+\frac{1}{m^2}\right) + \frac{m}{8}\left(1-\frac{1}{m^2}\right)^2 \sin^2\gamma + O(\sin^4\gamma)\right\} \tag{25}$$

whose first term is, remarkably, of the same form as (22) even though the reduction leading to (25) is entirely different. (25) provides in addition an estimate of the effect of γ on the correction curve. The term in $\sin\gamma$ is found to be small. Thus for $m=1.548$ we have

$$\zeta - \nu = \zeta(1.097 + 0.066\sin^2\gamma + \cdots)$$

For $\gamma=30°$ and $\zeta=30°$ the effect of the second term on $\zeta-\nu$ is 0.5°, in agreement with the value 0.5° found by actual calculation from the exact formula (20).

The approximate linear formula (22) turns out to fit the exact relation (20) rather well over the entire useful measurement range of standard universal stages, from $\zeta=0°$ to $\zeta=60°$. This is shown in Fig. 2, where the linear relation is plotted as a dashed line. Even for an index radio $m=1.548$, which is much larger than that normally encountered in universal stage work (except in ice petrofabrics), the linear curve departs from the exact curve for $\gamma=30°$ by more than 1° only above $\zeta=60°$, and below $\zeta=55°$ the curves do not depart by more than 0.5°. For values of m nearer to 1, the discrepancy between the linear and exact curves will be corresponding smaller, all curves converging for $m=1$ to the relation $\zeta-\nu=\zeta$, the curve of no correction. To a fairly good practical approximation it is thus possible to use (22) as the correction curve for the H setting over the entire useful measurement range of the universal stage and for any EW inclination γ likely to arise in practice. Only in the most exacting case, where an accuracy better than 0.5° is required, would correction have to be made by the exact relation (equations (16) to (20)).

The striking near-linearity of the correction curves given by (20) is not obvious in the formulas themselves, and therefore deserves a somewhat closer look. For the case $\gamma=0$, (20) reduces to the simpler form

$$\sin^2(\zeta-\nu)=\frac{2+m^2\tan^2\zeta+\sin^2\zeta-\cos\zeta\sqrt{1-\frac{1}{m^2}\sin^2\zeta}}{4+m^2\tan^2\zeta} \tag{26}$$

in which, however, the near-linearity of the $\zeta-\nu$ *vs.* ζ curve is still not apparent. The degree of non-linearity can be evaluated by first writing

$$\zeta-\nu=\sin^{-1}\sqrt{\sin^2(\zeta-\nu)} \tag{27}$$

where $\sin^2(\zeta-\nu)$ is given by (26), and then expanding (27) in powers of ζ. After considerable calculation this expansion leads to

$$\zeta-\nu=\zeta\frac{m}{2}\left(1+\frac{1}{m^2}\right)+\zeta^3\frac{(m^2-1)^2(2-m^2)}{24m^3}+O(\zeta^5) \tag{28}$$

The coefficient of the term in ζ^3 proves to be quite small, which accounts for the near-linearity of the correction curve. For $m=\sqrt{2}$ the third-order term vanishes exactly, and for $m=1.548$ it is $-0.0088\,\zeta^3$. At $\zeta=40°$ the value of this term is $-0.2°$, which agrees approximately with the difference $-0.3°$ between the exact curve calculated from (20) and the linear curve from (22). At higher ζ values the deviation of the exact curve from the linear curve is rather larger than given by the third-order term in (28), which doubtless indicates the influence of the higher-order terms. Nevertheless (28) gives a good idea of the order of magnitude of the non-linearity of the H-setting correction curve; for $m=1.548$ the curvature of the Snell's law curve is about 12 times greater.

The above considerations justify using (22) as the basis for discussing the general features of correction curves given by (16)–(20). Solved for ν, (22) becomes

$$\nu=-\zeta\frac{(m-1)^2}{2m} \tag{22a}$$

From (22*a*) it is seen that ν is always negative. For $m>1$ this means that the true correction ν is in the opposite direction to that given by Snell's law. For $m<1$ (refractive index of crystal plate less than that of hemispheres) the Snell's law and true corrections are in the same direction, but the shapes of the two correction curves are quite different. The magnitude of the true correction is always much less than that given by Snell's law. In fact, for most practical purposes where hemispheres are used, the correction given by (22a) is negligibly small. Thus if hemispheres of refractive index 1.51 and 1.65 are available, as normally, then for crystals of refractive index in the range 1.45 to 1.74 the index ratio m need not lie outside the range 0.95 to 1.05. For $m=0.95$ and $m=1.05$ we find for $\zeta=60°$ a true correction ν of only $-0.08°$, from (22a), whereas the Snell's law correction for the same tilt ζ and index ratios m is $\pm 5.0°$.

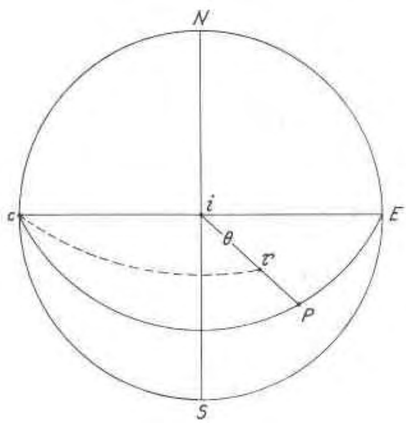

FIG. 3

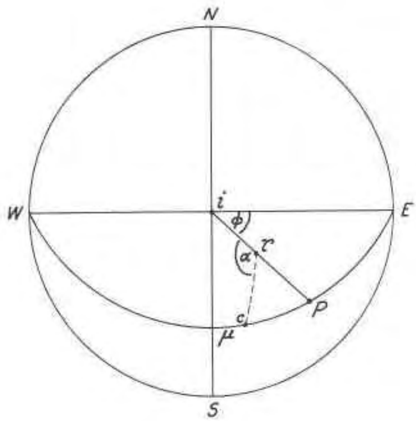

FIG. 4

FIG. 3. Stereographic projection as in Fig. 1, showing the situation when $\nu=0$. In this diagram θ is shown larger than in Fig. 1, for greater clarity in illustrating the situtation at r.

FIG. 4. Stereographic projection for analysis of refraction-correction of universal stage orientation measurement of a uniaxial crystal in V setting. Notation is in Fig. 1. μ is the correction that must be subtracted from the measured tilt ζ to get the true inclination $\zeta-\mu$ of the c-axis to the normal to the thin section.

The fact that Snell's law fails for the H setting, and the fact that for $m>1$ the true crystal orientation differs from the measured orientation in the direction opposite to that toward which the light is refracted in the crystal, are phenomena that defy ordinary intuition in crystal optics. They can, however, be understood qualitatively by reference to Figs. 1 and 3. If in Fig. 1 we seek to imagine the position that c must take, for a given fixed P and r, to satisfy the required condition that the tangent to the great circle cr at r be parallel to WiE, then evidently c must move toward W so that ν decreases from the size shown in the figure. When c reaches W, so that $\nu=0$, the situation is as shown in Fig. 3. It is clear in Fig. 3 that c must continue to move even further in the same direction, beyond W, before the required condition at r becomes satisfied, so that ν must become negative. By the same type of argument, the fact that for $m<1$, ν is again negative can also be understood. In this case r lies on the opposite side of i from P.

Uniaxial Crystals in V Setting

In the V setting, at minimum transmitted light c lies somewhere in the neighborhood of the NiS plane, as shown in Fig. 4. A correction μ is to be applied to the measured tilt ζ on NS in order to get the true inclination $\zeta-\mu$ of the c-axis to the normal to the crystal plate. The analysis of this case follows similar lines to that of the H setting. If we recognize that

$$\mu = \nu - \frac{\pi}{2} \tag{29}$$

then equations (1) to (8) can be applied directly, except that the extinction condition in (3) must be replaced by

$$\alpha = \phi + \frac{\pi}{2} \tag{30}$$

Combining equations (1), (2), (4)–(8), (29), and (30) by the same procedure as before, and eliminating the extraneous roots by the same method, we obtain the result

$$\cos(\zeta - \mu) = +\sqrt{f^{+}(\omega, \sigma, \Lambda)} \tag{31}$$

where $f^{+}(\omega, \sigma, \lambda)$ is given in (19). ω and σ are given by (16) and (17), as before, but for the V setting λ is replaced by

$$\Lambda = \frac{\sin^2\gamma}{\sin^2\zeta} - 1 \tag{32}$$

The first step in evaluating (31) is to examine the case $\gamma=0$. It might be expected that any value of μ should satisfy (31) in this case, because when the stage is not tilted on EW the crystal remains at extinction for any tilt ζ on NS. However, although this expectation is substantiated in the form of the original equations, it has been eliminated in the reduction leading to (31), and thus (31) evaluated for $\gamma=0$ gives the limiting correction curve $\zeta-\mu$ *vs.* ζ as $\gamma\rightarrow 0$, *i.e.* for very small tilt angles on EW. With $\gamma=0$, (31) reduces exactly to

$$\sin(\zeta - \mu) = \frac{1}{m}\sin\zeta \tag{33}$$

which is the ordinary Snell's law correction formula.

To investigate the effect of γ on the correction curve, we could make a detailed study along the lines given for the H setting, but because in practice only small tilt angles on EW are necessary, it will suffice here to examine only the numerical evaluation of the particular case $m=1.548$. In Fig. 2 is plotted the V-setting correction curve for $\gamma=0°$ (labelled *Snell's law*), from (33), and also the curve for $\gamma=40°$, calculated from (31). The effect of tilting on EW becomes appreciable only at high ζ. For $\gamma=20°$ the theoretical correction curve (31) departs less than 0.1° from the Snell's law curve (33) over the range $0° \leq \zeta \leq 40°$, and at $\zeta=60°$ the departure is only 0.4°. In practice it is never necessary to use as high an EW tilt as 20°, so that evidently the curve (33) is quite adequate for any V-setting measurement of the accuracy normally required.

In the same way that we can understand why the Snell's law correction curve fails for the H setting, we can also see why it should hold approximately in the V setting. Perhaps the best way to express this is to say that the uniaxial optic axis interference figure is very nearly a perfect cross, and the extinction condition is satisfied when in Fig. 4, thought of as an interference figure, the NS arm of the cross from c passes through r. It is also clear by the same argument that as $\gamma \rightarrow 0$ the Snell's law correction must become exactly correct over the entire range of ζ. The main contribution of the detailed analysis, leading to (31), is thus that it allows the effect of non-zero γ to be precisely allowed for in the unusual case in which this happens to be necessary.

Experimental Test with Quartz

The most sensitive test of the above theory is obtained by choosing a value of m that differs as much as possible from unity. I have therefore made measurements on a universal stage without hemispheres, which is the same as using hemispheres of index unity, so that $m=n_{xl}$ (see equation (2)). The thin sections used were quartz, cut at various angles to the optic axis. The sections were mounted on the inner-stage platform by means of an oil nearly matched to the refractive index of the platform, so that the sequence of light refractions below and above the quartz would be essentially the same. The apparent inclination (tilt angle ζ on the NS axis) for the c-axis of each section, in H or V setting as appropriate, was measured first without hemispheres and second with hemispheres of index $n_s=1.516$, which differs from the mean quartz index 1.548 by relatively little, so that from the latter measurements the true inclination $\zeta-\nu$ or $\zeta-\mu$ could be obtained with only slight refraction-correction (as mentioned above, for the H setting the correction in this case is negligible and for the V setting the Snell's law correction is to be used). The resulting measured ζ and $\zeta-\nu$ or $\zeta-\mu$ values can then be compared directly with the theoretical curves for $m=1.548$ given above.

Each inclination (both with and without hemispheres) was measured twenty times for a given fixed setting of the IV axis, ten times with the stage tilted on the *EW* axis by a given amount γ toward the observer and ten times tilted on *EW* by the same amount γ away from the observer. The reason for this procedure is that the two values ζ_S (for tilt on *EW* toward observer) and ζ_N (tilt away) when determined precisely by repeated measurement in general differ significantly, often by more than one degree. The discrepancy is due to a slight error in the initial extinction-setting on *IV*, made before tilting on *EW* and *NS*.[1] Because the measured value ζ is rather sensitive to the *IV* setting, and because the effect on ζ_N of a small change in *IV* is equal and opposite to the effect on ζ_S, a slight error in the *IV* setting can produce a pronounced discrepancy between ζ_N and ζ_S, and it is difficult to set *IV* so accurately that the discrepancy altogether disappears. As an example, it is found experimentally that for a quartz section with *c*-axis inclined 32.7° to the plane of the thin section, the dependence of the measured angles ζ_N and ζ_S on *IV* setting ϕ, under measurement without hemispheres, is $d\zeta_N/d\phi = +1.7$ and $d\zeta_S/d\phi = -1.7$, so that an error of only 0.5° in the *IV* setting produces a 1.7° discrepancy between the ζ_N and ζ_S values. Since for the present purpose there is no reason to try to determine the true *IV* setting very accurately, and since for small errors in *IV* the sum $\zeta_N + \zeta_S$ remains constant, the true value ζ was determined by measuring ζ_N and ζ_S separately and taking

$$\zeta = \tfrac{1}{2}(\zeta_N + \zeta_S)$$

The above procedure, in addition to being time-saving, improves the measurements statistics over what would be obtained if *IV* were independently reset before each ζ measurement. The standard deviations of the ζ_N and ζ_S measurements can be determined separately and the estimated standard deviation of the average value calculated in the usual way. The estimated standard deviations of the final mean ζ and $\zeta - \nu$ values, as so determined, range from 0.10° to 0.20°.

For the measurements in *H* setting a fixed γ value of 30° was used, and for the *V* settings 15°.

The results of these measurements are plotted in Fig. 5, where they can be compared with the theoretical *H* setting correction curve for $\gamma = 30°$ and with the *V* setting correction curve for $\gamma = 0°$ (Snell's law), the latter differing only insignificantly from the curve for $\gamma = 15°$ over the range for which measured points are available. The agreement between measurements and the theoretical curves is satisfactory, although the scatter of the points is rather larger than can be expected from the

[1] A similar discrepancy is also produced by slight misalignment of the *NS* and *EW* stage axes with respect to the *NS* and *EW* directions defined by polarizer and analyzer; the resulting error is also eliminated by the averaging procedure described.

precision of measurement alone. The standard deviation of the ζ values, estimated from the scatter of the measured values about the theoretical curves, is 0.5°, whereas from the statistics of the actual measurements the estimated standard deviation is about 0.15°. The measurements appear to be affected by small, unpredicable systematic errors—possibly due to systematic personal error in judging the minimum-light setting or to slight differences in the sequence of refracting surfaces above and below the quartz plates. For the most exacting measurements of crystal orientation, with accuracy of 0.1°, the source of these errors would have to be found. Nevertheless, the correctness of the foregoing theory is confirmed by the measurements.

Experimental Test with Ice

Because the index-of-refraction corrections in ice petrofabric work are doubtless the largest that ever arise in practice, it is desirable to have a

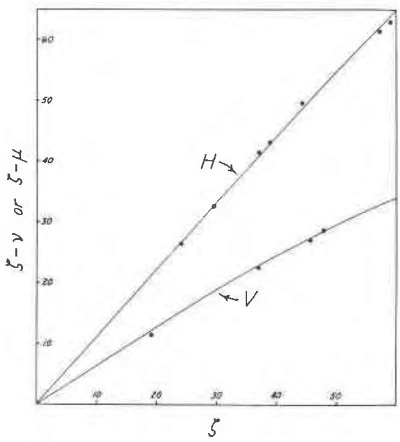

Fig. 5

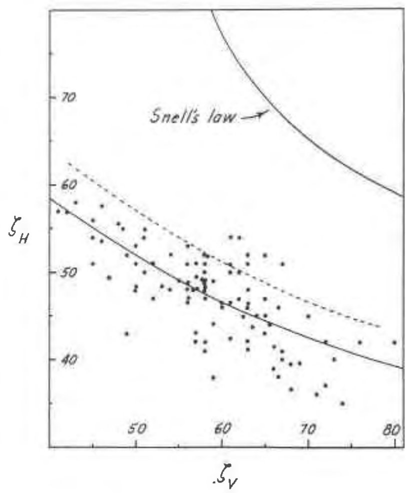

Fig. 6

Fig. 5. Comparison of theoretical refraction-correction curves for quartz ($n=1.548$) with measurements on quartz thin-sections. The H curve and measurements are for $\gamma=30°$. The V curve is for $\gamma=0°$, although the points were measured using $\gamma=15°$. For each measured point, $\zeta-\nu$ (or $\zeta-\mu$) is the true inclination, measured using hemispheres of index $n=1.516$, and ζ is the apparent inclination as measured with hemispheres absent.

Fig. 6. Measurements on ice crystals that can be oriented both in H and V setting. The tilt angle (on the NS universal stage axis) measured in the V setting is plotted for each crystal against the tilt angle measured in the H setting. The upper solid curve is the expected relationship of these angles based on the Snell's law correction, while the lower solid curve is calculated from the present theory. The dashed curve is based on the empirical correction curve given by Langway (1958).

direct demonstration of the applicability of the above theory to ice, even though, because of the greater difficulty of handling the material and the greater inaccuracies of measurement, the test of the theory in this case cannot be as rigorous as with quartz.

A first test can be made against Rigsby's measurements, reported briefly by Langway (1958, p. 7). Langway gives an empirical correction curve for the H setting that is a straight line with slope $(\zeta-\nu)/\zeta=1.12$. The corresponding slope given by equation (22) (for $m=1.31$, the mean refractive index of ice) is 1.039. This slope is significantly smaller than that of Langway's curve. The discrepancy is not the result of the effect of tilting on the EW axis of the universal stage, because from (25) the slope is increased only to 1.044 for $\gamma=30°$, about as large a value as is likely to be used in practice. Curvature of the correction curve is also not responsible, because according to (28) the curvature is less than one third as great as for quartz, although for ice the curvature is positive rather than, as for quartz, negative. Without having Rigsby's actual measured points available it is impossible to comment further on this discrepancy. Qualitatively, however, the present theory and Rigsby's measurements are in agreement in indicating a linear or nearly linear correction curve of slope greater than unity.

A second approach is to make use of ice crystals whose c-axes are tilted about 45° to the plane of the thin section, and for which, as mentioned above, it is often possible to measure the c-axis orientation both in H setting and in V setting. Over a period of four years I have collected 95 such pairs of measurements in the course of ice petrofabric work, made with two different universal stages. In Fig. 6 the actual measured values are shown by plotting, for each pair of measurements, the tilt on NS in the H setting (ζ_H) against the tilt in the V setting (ζ_V) for the same crystal. According to the present theory, assuming for simplicity the H-setting correction curve of (22), these measurements should be related by

$$\zeta_H = \frac{1}{1.04}\left(90° - \sin^{-1}\frac{\sin\zeta_V}{1.31}\right) \tag{34}$$

whereas according to the Snell's law correction for both H and V settings the relation should be

$$\zeta_H = \sin^{-1}\left[1.31\sin\left(90° - \sin^{-1}\frac{\sin\zeta_V}{1.31}\right)\right] \tag{35}$$

These two curves are plotted in Fig. 6 (solid curves). Also shown is the corresponding curve (dashed) based on the empirical correction curve for the H setting given by Langway. It is the same as (34) with the factor 1.04 replaced by 1.12.

The scatter of the measured values in Fig. 6 is very large, which reflects the large errors in such measurements under field conditions. Nevertheless, the Snell's law curve is obviously ruled out, and it is seen that the curve based on the present theory fits the measurements definitely better than that based on Langway's empirical curve. Within the rather wider errors, these measurements confirm the present theory.[1]

It is thus justified in ice petrofabric work (without hemispheres) to correct the measured inclinations in H setting by the following simple relation:

True inclination of c-axis to plane of thin section $= 1.04 \times$ *measured tilt on NS axis.*

Because of the many other sources of inaccuracy in such measurements it does not seeem worthwhile to apply the more accurate correction based on equations (16)–(20), which differs only very slightly from the above linear relation. Measurements in V setting are to be corrected by Snell's law in the usual way.

Biaxial Crystals

It is evident that the theoretical approach used here can be extended to biaxial crystals. Although I have not done this, the above experience with uniaxial crystals makes it possible to comment to some extent on the expected results for biaxial crystals.

For biaxial crystals the type of refractive-index correction will depend on the optical element whose orientation is measured. A primary distinction will have to be made between (1) measurements in which an optic axis is set "vertical" and (2) measurements in which an indicatrix symmetry plane is set "vertical" or in which a principal axis is set "horizontal" or "vertical." For measurements of type (1) the Snell's law correction will doubtless be valid to a good approximation, whereas for type (2) it will doubtless fail to a greater or lesser extent. If the measurement places the acute bisectrix in the H setting (ostensibly parallel to

[1] Since this was written I have made the same type of measurements on ice in thin section under the microscope, in connection with experimental work on ice recrystallization at the Eidgenössisches Institut für Schnee- und Lawinenforschung, Weissfluhjoch, Davos, Switzerland. The microscopically measured (ζ_V, ζ_H) values cluster closely about the theoretical curve (34), verifying it to a degree of accuracy ($\pm 0.5°$) considerably greater than is possible with the field measurements shown in Fig. 5. In the laboratory ice petrofabric work, where hemispheres could have been used, I found it much more convenient in handling and interchanging thin sections to work without hemispheres and to use the refraction correction curves from the present paper. This experience suggests that in ordinary routine petrofabric work, as with quartz, it may also be convenient to dispense with hemispheres, especially when many thin sections must be measured. The refraction corrections can be combined with the radial scale-function of the Schmidt net to construct a plotting scale with which the measured inclinations can be plotted directly on the net or projection, avoiding the need to apply the corrections numerically.

EW), then the situation is much like that of a uniaxial crystal in *H* setting, and a nearly linear correction curve similar to (22) will doubtless be valid. Although the slope of the correction line cannot be predicted without detailed analysis, it seems likely that the correction ν will vary as $(m-1)^2$, as in (22*a*), and that as in the uniaxial case the correction will be very small in most practical situations with standard hemispheres. Without an exact theory the best thing to do in this case is to assume that the measured and true inclinations are the same; a Snell's law correction would be much further from the truth. The same situation may apply to measurements in which the orientation of the obtuse bisectrix is measured in the *H* setting, although it seems likely that the correction will be significantly different depending on whether the optic plane is more nearly parallel to the plane of the crystal plate or perpendicular thereto. Finally, measurements of the optic normal in *H* setting appear to pose greater complications, and it even appears possible that a refraction-correction significantly greater than that given by Snell's law may be required under certain conditions. Theoretical and experimental study of these various cases is clearly desirable.

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