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TRANSFORMATION OF TRILINEAR AND QUADRIPLANAR COORDINATES TO AND FROM CARTESIAN COORDINATES[1]

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ABSTRACT

Trilinear and quadriplanar coordinates are types of homogeneous coordinates that have many applications in mineralogy and geochemistry, but the analytic geometry of such coordinates is not generally known. It therefore happens that the data derived by these systems are treated graphically rather than analytically. An analytical treatment can be obtained without recourse to unfamiliar analytic geometry by the direct and reverse transformation of trilinear and quadriplanar coordinates to cartesian coordinates, thus resolving all such problems to conventional analytic geometry.

Formulae for the necessary transformations have been published in generalized form to apply to triangles and tetrahedra of all kinds, with the origins of cartesian coordinates variously placed inside these figures. Formulae with numerical constants, however, have not been published for the simplest figures, with specifically placed origins, as required in practical applications. Such transformations are the principal thesis of this paper.

The equilateral triangle is used as a triangle of reference for trilinear coordinates, and the regular tetrahedron as a tetrahedron of reference for quadriplanar coordinates. The origins of 2-dimensional and 3-dimensional cartesian coordinates are placed at the centroids of these figures. Three transformation formulae are deduced for trilinear coordinates, and four such formulae are derived for quadriplanar coordinates. Some suggestions are made regarding a generalization of quadriplanar and 3-dimensional cartesian coordinates into the fourth dimension, but no transformation formulae are presented. Practical applications are outlined for homogeneous and cartesian coordinates.

INTRODUCTION

The subject of trilinear coordinates is omitted from modern American textbooks on analytic geometry, but has been fully developed in older British treatises. Among these are the works of Whitworth (1866, 506 p.), Ferrers (1876, 184 p.), Loney (1923, vol. 2, 288 p.), and a concise treatment by Smith (1919, p. 341–377). Quadriplanar coordinates have been discussed by Snyder and Sisam (1914, p. 109–123) under the heading of tetrahedral coordinates, but this designation is misleading, as tetrahedral coordinates are volumetric, referring to the ratios of the volumes of four minor tetrahedra within a tetrahedron of reference. Quadriplanar coordinates are also discussed in books on projective geometry, mainly in connection with general homogeneous coordinates and equations.

Trilinear and quadriplanar coordinates, for reasons later stated, may need to be transformed into 2-dimensional and 3-dimensional cartesian coordinates; and the reverse transformations are equally necessary. Trilinear coordinates constitute a system of analytic geometry whereby points and curves are located in relation to a triangle of reference. Any

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triangle may be used for this purpose; and in a transformation to cartesian coordinates, the origin may be placed anywhere within the selected triangle. Generalized formulae for such transformations are given in the books cited above, and also in the volume by Carr (1886, p. 562); but specialized formulae with numerical constants for simplified calculations have not been published. Two limitations are necessary. It is obvious that an equilateral triangle is the simplest triangle of reference; and the logical and most practical placement for the origin of 2-dimensional cartesian coordinates is at the centroid of the equilateral triangle. With these two restrictions, simple transformation formulae may readily be developed.

Generalized formulae have also been published for the transformation of quadriplanar coordinates to 3-dimensional cartesian coordinates, and vice versa. But no simplified transformation formulae with numerical constants for practical calculations have been made available. Points, space curves, and surfaces are located in quadriplanar coordinates with regard to a tetrahedron of reference. Such a tetrahedron may have triangular faces of various shapes, but the simplest figure is that of a regular tetrahedron with equilateral triangular faces. This is selected as the tetrahedron of reference; and the origin of 3-dimensional cartesian coordinates is placed at the centroid of the tetrahedron.

One or two trilinear coordinates may be negative, but not three. If one coordinate is negative, the corresponding point lies outside one of the edges of the triangle of reference; but if two coordinates are negative, the point lies within an exterior angle of the triangle. Points in quadriplanar coordinates may have one, two, or three negative values. If one coordinate is negative, the corresponding point lies outside one of the faces of the tetrahedron of reference; if two coordinates are negative, the point lies within an exterior dihedral angle of the tetrahedron; and if three coordinates are negative, the point lies within an exterior trihedral angle of the tetrahedron.

Another limitation exists regarding negative values for trilinear and quadriplanar coordinates. If the sum of two negative trilinear coordinates equals or exceeds the value of the positive coordinate, or if one negative trilinear coordinate equals or exceeds the sum of two positive coordinates, no point can be charted. Similarly, if one or the sum of two or three quadriplanar coordinates equals or exceeds the value of the positive coordinate or coordinates, no chartable point exists.

Trilinear Coordinates

The triangular coordinate paper (no. 358-32) furnished by Keufel and Esser, New York, is built upon an equilateral triangle, and exemplifies the form and calibration commonly used in charting trilinear coordinates. An equilateral triangle of reference is shown in Fig. 1, with the origin of 2-dimensional cartesian coordinates at the centroid of the triangle, and with the X-axis parallel to the base of the triangle. Also shown are the trilinear coordinates α, β, and γ of a point P, and the angles ω_1, ω_2, and ω_3, which these coordinate lines make respectively with the X-axis. By

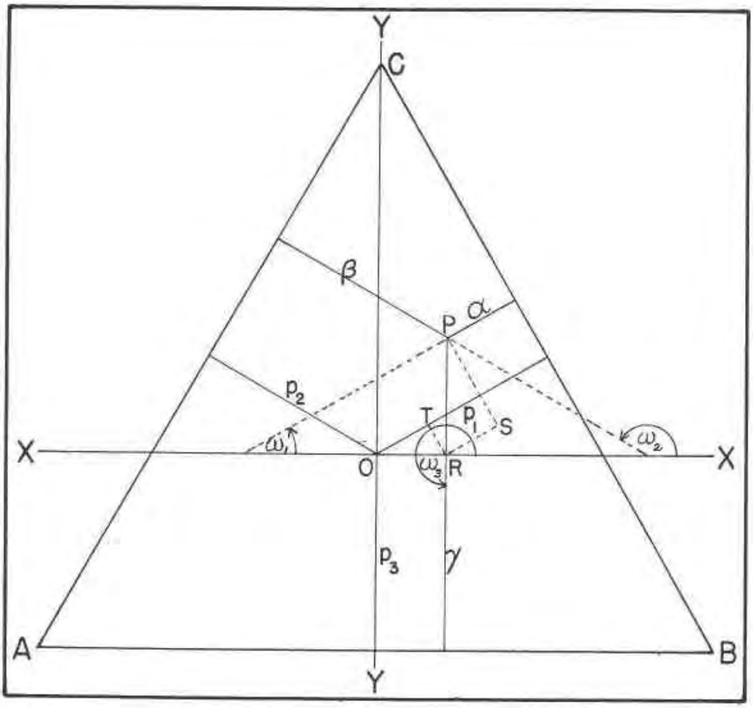

FIG. 1. Relationship between trilinear and 2-dimensional cartesian coordinates.

drawing the construction lines PS, SR, and RT, the length of the coordinate alpha is so self-apparent that no formal proof is needed; and the same relationships hold with regard to beta and gamma. Hence, on the assumption that the sum of the trilinear coordinates is 100, the transformation formulae are readily seen to be the following:

$$\alpha = p_1 - x \cos \omega_1 - y \sin \omega_1 \quad (1)$$

$$\beta = p_2 - x \cos \omega_2 - y \sin \omega_2 \quad (2)$$

$$\gamma = p_3 - x \cos \omega_3 - y \sin \omega_3 \quad (3)$$

where

$$p_1 = p_2 = p_3 = 33\tfrac{1}{3} \quad \omega_1 = \frac{\pi}{6}, \omega_2 = \frac{5\pi}{6}, \text{ and } \omega_3 = \frac{3\pi}{2}$$

In their simplest form, equations (1), (2), and (3) may be written as follows:

$$\alpha = 33.3333 - .8660x - .5y \tag{4}$$

$$\beta = 33.3333 + .8660x - .5y \tag{5}$$

$$\gamma = 33.3333 + 0 + y \tag{6}$$

By means of these formulae, 2-dimensional cartesian coordinates may quickly be transformed to trilinear coordinates. In the reverse transfor-

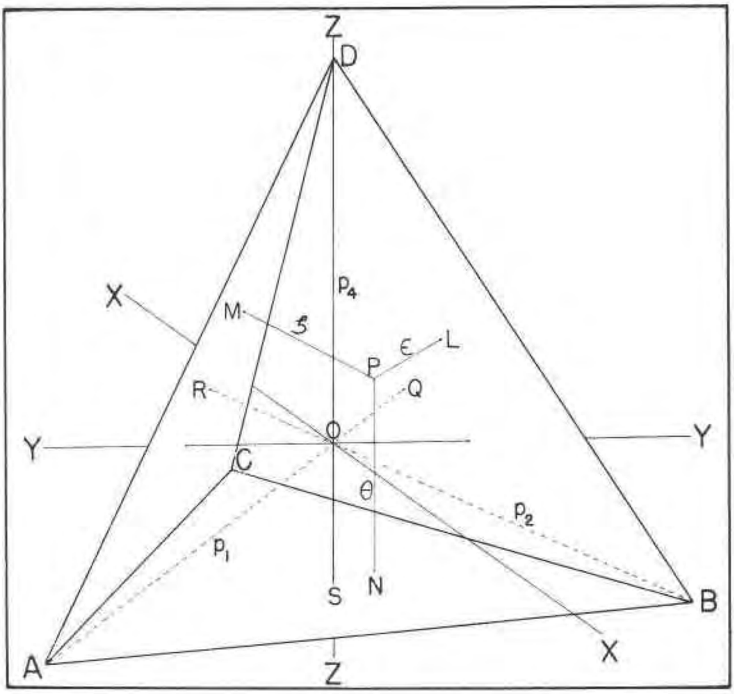

FIG. 2. Perspective view of a regular tetrahedron with the face ABD removed. Illustrates relationship between quadriplanar and 3-dimensional cartesian coordinates.

mation, the value of y is obtained from formula (6); and the value of x may be obtained either from formula (4) or (5), as these three formulae are consistent.

QUADRIPLANAR COORDINATES

The relationship between quadriplanar and 3-dimensional cartesian coordinates is somewhat more involved, but the final formulae that are deduced are readily applied. Figure 2 shows a perspective view of a regular tetrahedron of reference, with the front face removed. The origin of

3-dimensional cartesian coordinates is made coincident with the centroid of the tetrahedron. The Y-axis is parallel to the edge AB, and the Z-axis is congruent with the normal from D to the face ABC. A simple trigonometric computation shows that the four normals from the vertices to the four opposite faces, exemplified by AQ, BR, and DS, meet in a point that is the centroid of the tetrahedron. (This is true only for a regular tetrahedron.) Another calculation shows that each of the segments OQ, OR, and OS, as well as the distance from O to the face ABD, is equal to 25, if the sum of the quadriplanar coordinates is 100.

A concise proof is given by Yates (1961, p. 190–191) that the distance of a point (x, y, z) to a plane, exemplified by any of the four faces of a tetrahedron, is represented in cartesian coordinates by the following equation:

$$s = x \cos \alpha + y \cos \beta + z \cos \gamma - p \qquad (7)$$

where

s = the length of a directed line from P to the given plane.
x, y, and z = the cartesian coordinates of P.
α, β, and γ = the direction angles (not the trilinear coordinates) of the normals AQ, BR, DS, and a similar normal from C to the face ABD.

and

p = the length of each of the segments OQ, OR, OS, and a similar segment from O to the face ABD.

The directed distances in quadriplanar coordinates, however, are measured in the opposite direction, that is, from the four faces of the tetrahedron of reference to the given point. Hence the algebriac signs of the right side of equation (7) must be reversed in order to show the proper relationship, as given below.

$$s_i = p_i - x \cos \alpha_i - y \cos \beta_i - z \cos \gamma_i \; (i = 1, 2, 3, 4) \qquad (8)$$

Equation (8) is found to be the one required for the transformation of 3-dimensional cartesian to quadriplanar coordinates, and vice versa. Hence the quadriplanar coordinates of a set of 3-dimensional cartesian coordinates may be written as follows:

$$\epsilon = p_1 - x \cos \alpha_1 - y \cos \beta_1 - z \cos \gamma_1 \qquad (9)$$

$$\zeta = p_2 - x \cos \alpha_2 - y \cos \beta_2 - z \cos \gamma_2 \qquad (10)$$

$$\eta = p_3 - x \cos \alpha_3 - y \cos \beta_3 - z \cos \gamma_3 \qquad (11)$$

$$\theta = p_4 - x \cos \alpha_4 - y \cos \beta_4 - z \cos \gamma_4 \qquad (12)$$

It has already been stated that $p_1 = p_2 = p_3 = p_4 = 25$. The direction angles and cosines of the four coordinate lines are the same as those of the normals from the four vertices to the opposite faces. It remains, therefore, to determine the magnitudes of these 12 direction angles and their cosines. The direction angles have been found to have the following values.

Direction angles

$\alpha_1 = 118°\ 7'\ 31.8''$	$\beta_1 = 35°\ 15'\ 51.8''$	$\gamma_1 = 70°\ 31'\ 43.6''$
$\alpha_2 = 241°\ 52'\ 28.2''$	$\beta_2 = 144°\ 44'\ 8.2''$	$\gamma_2 = 70°\ 31'\ 43.6''$
$\alpha_3 = 19°\ 28'\ 16.4''$	$\beta_3 = \frac{\pi}{2}$	$\gamma_3 = 70°\ 31'\ 43.6''$
$\alpha_4 = \frac{\pi}{2}$	$\beta_4 = \frac{\pi}{2}$	$\gamma_4 = \pi$

Direction cosines

$\cos\alpha_1 = -.4714$	$\cos\beta_1 = .8165$	$\cos\gamma_1 = .3333$
$\cos\alpha_2 = -.4714$	$\cos\beta_2 = -.8165$	$\cos\gamma_2 = .3333$
$\cos\alpha_3 = .9428$	$\cos\beta_3 = 0$	$\cos\gamma_3 = .3333$
$\cos\alpha_4 = 0$	$\cos\beta_4 = 0$	$\cos\gamma_4 = -1.0000$

Hence, in their simplest form, the formulae for transforming 3-dimensional cartesian to quadriplanar coordinates and vice versa, are the following:

$$\epsilon = 25 + .4714x - .8165y - .3333z \quad (13)$$

$$\zeta = 25 + .4714x + .8165y - .3333z \quad (14)$$

$$\eta = 25 - .9428x + 0 - .3333z \quad (15)$$

$$\theta = 25 + 0 + 0 + z \quad (16)$$

In the reverse transformation from quadriplanar to 3-dimensional cartesian coordinates, the value of z is obtained directly from equation (16); and by substituting in equation (15), the value of x emerges. As these four equations are consistent, the value of y may be computed either from equation (13) or equation (14).

Pentahyperflat Coordinates

Five variables, of which four are independent, are theoretically chartable in what may be described as pentahyperflat coordinates, which are referred to a regular hypertetrahedron of the fourth dimension. Such a polytope, called a regular simplex, has 5 vertices, 10 edges, 10 equilateral triangular faces, and 5 bounding hyperflats (which are in fact regular tetrahedra). The coordinates are parallel to the 5 normals from the vertices to the hyperflats. As the available data cannot be charted in four dimensions, they may be projected either onto the triangular faces or onto the bounding hyperflats (tetrahedra). Charts designed for these purposes have been prepared by Mertie (1948, p. 324–336, and 1949, p. 706–716).

The analytic transformation of five pentahyperflat coordinates to four 4-dimensional cartesian coordinates, and vice versa, might be useful. In particular, the elimination of one variable in mathematical analysis should be an important asset. The necessary transformation formulae, however, are not known to the writer. If they are analogous to those used for quadriplanar coordinates, five formulae are needed, and 20 direction angles must be determined.

Applications

Trilinear and quadriplanar coordinates are species of homogeneous coordinates that are characterized by two or three independent variables and one dependent variable. Such coordinates add to 100, or by recomputation can be made to do so. The transformation formulae so far developed make it possible to transform trilinear and quadriplanar coordinates to cartesian coordinates, and vice versa; but in this process no summation of the cartesian coordinates is involved.

Originally determined cartesian coordinates differ from homogeneous coordinates in that all the coordinates are independent variables. Therefore it would be quite improper to recompute a set of cartesian coordinates to total 100, with the objective of utilizing them in the preceding transformation formulae. The result would be the derivation of number sets that would not be uniquely distinctive of the original coordinates. Thus the cartesian coordinates (10, 15, 25) and (20, 30, 50), whose sums are respectively 50 and 100, represent two distinct 3-dimensional points. Yet with certain modifications, formulae (4)–(6) and (13)–(16) may be used to handle cartesian coordinates homogeneously. The charting and analysis of homogeneous coordinates and independent cartesian coordinates are separately discussed in the following pages.

Homogeneous Coordinates

Experimental results of numerous mineralogical and geochemical investigations, as well as analyses with three terms, are plotted in trilinear coordinates; but analytical work on the points and curves thus charted is rarely done, doubtless because the analytic geometry of trilinear coordinates is not generally understood. Hence general relationships that could be expressed by algebriac equations are not obtained. But if the desired results could be obtained without recourse to this specialized analysis, they probably would be acceptable. Such results can be gotten by the use of formulae (4), (5) and (6). Trilinear points may obviously be transformed to 2-dimensional cartesian coordinates, and joined graphically or by analytic fitting to form smooth curves; and trilinear curves, using a point-by-point technique, may be similarly transformed. The resulting cartesian curves may then be differentiated, integrated, or subjected to any other analytical process, after which they may be transformed back to trilinear coordinates. Thus the analytical geometry of trilinear coordinates may be entirely avoided. It is true that the algebriac results of such work will be available only in cartesian coordinates; but for most workers and readers this will probably be an asset rather than a drawback.

Analyses with three terms, that chart naturally in trilinear coordinates, may likewise be reduced to 2-dimensional cartesian coordinates. The two resulting variables will have different meanings than the original three, but the gain in simplicity of charting may under some circumstances be desirable. In reverse, two variables that were derived originally as cartesian coordinates may be transformed into trilinear coordinates, if that should be advantageous.

Examples of the transformation of trilinear to 2-dimensional cartesian coordinates may be given by a mean analysis of beryl, derived from 11 analyses taken from Dana (1914, p. 407); and by a mean analysis of olivine (chrysolite), derived from 32 analyses from the same source (1914, p. 453). These two analyses, recomputed free of minor impurities to total 100 per cent, are as follows.

	Beryl	*Olivine*
SiO_2	67.83	40.46
Al_2O_3	18.84	—
FeO	—	13.96
MgO	—	45.58
BeO	13.33	—
Total	100.00	100.00

From formulae (4), (5), and (6), the values of the cartesian coordinates are found to be the following:

Beryl $(x, y) = (-28.28, -20.00)$ Olivine $(x, y) = (-15.30, 12.25)$

Many mineralogical, geochemical, and geophysical investigations yield four variables, of which three are independent of one another. These are seldom charted directly, as this operation would require perspective drawings of a tetrahedron of reference. Instead, the points, space curves, or surfaces are projected, either apically or orthogonally, onto the triangular faces of the tetrahedron, which is then developed as four triangles. Or, if charting is not attempted, the numerical results are merely tabulated. Such tetrahedral data can readily be transformed into 3-dimensional cartesian coordinates by the use of formulae (13), (14), (15) and (16). Thereafter the analytical methods heretofore suggested may be applied, and subsequently reverse transformations may be made. This reduces all problems in quadriplanar coordinates to conventional solid analytical geometry.

Analyses with four terms, of which three are independent of one another, may similarly be transformed into 3-dimensional cartesian coordinates, if that should be desirable. The new points may then be located graphically in three dimensions; or if a number of such points were to be compared, they could be joined as space curves or surfaces, either by graphical or analytical treatment.

The transformation of quadriplanar to 3-dimensional cartesian coordinates may be exemplified by a mean analysis of orthoclase, taken from 23 analyses published by Dana (1914, p. 319); and by a mean analysis of microcline, derived from 19 analyses from the same source (1914, p. 323–324). These two analyses, recomputed free of minor impurities to total 100 per cent, are as follows:

	Orthoclase	*Microcline*
SiO_2	65.57	65.18
Al_2O_3	19.43	19.82
Na_2O	4.06	2.30
K_2O	10.94	12.70
Total	100.00	100.00

From formulae (13), (14), (15), and (16), the values of the corresponding cartesian coordinates are found to be the following:

$$\text{Orthoclase, } (x, y, z) = (27.18, \ -28.25, \ -14.06)$$
$$\text{Microcline, } (x, y, z) = (28.43, \ -27.78, \ -12.30)$$

Cartesian Coordinates

Four cartesian coordinates are now presented to illustrate independent variables that may not be recomputed to total 100 per cent. These variables, which range in value from zero to infinity, represent the class, order, rang, and subrang of analyzed rocks in the C.I.P.W. classification of igneous rocks. The example is a biotite granite from El Capitan, Yosemite Valley, California, whose analysis and molecular ratios are tabulated by Washington (1917, p. 185).

Constants of biotite granite

Class	19.68	
Order	2.34	
Rang	2.13	
Subrang	0.75	
Total	24.90	whence $\frac{24.90}{4} = 6.225$

It will be noted in formulae (13), (14), (15), and (16) that $p_i = 25$. In order to treat this analysis as if its variables were homogeneous, it is only necessary to substitute the values of $p_i = 6.225$. This is equivalent to imagining that the point represented by this analysis is charted in a tetrahedron whose normals from the vertices to opposite faces have a length of 24.90. This treatment also implies that one of the four variables is dependent on the other three, which is not true, but is warranted as an empirical procedure. The resulting coordinates are as follows:

$$\begin{aligned} 19.68 &= 6.225 + .4714x - .8165y - .3333z \\ 2.34 &= 6.225 + .4714x + .8165y - .3333z \\ 2.13 &= 6.225 - .9428x + 0 - .3333z \\ 0.75 &= 6.225 + 0 + 0 + z \end{aligned}$$

The desired values of the cartesian coordinates are

$$(x, y, z) = (6.279, \ -10.618, \ -5.475).$$

Now using the unmodified form of formulae (13), (14), (15) and (16), it is possible to obtain a set of quadriplanar coordinates. These are found to have the following values.

$$(\epsilon, \zeta, \eta, \theta) = (38.46, 21.12, 20.90, 19.52)$$

Thus the C.I.P.W. constants may be expressed either as three cartesian coordinates, or as four quadriplanar coordinates, and may therefore be charted.

The question may be raised whether the empirical operation applied to the C. I. P. W. constants may not be repeated for the three derived cartesian coordinates, in order to reduce their number to two. Obviously this may not be done for the values of (x, y, z) shown above, because if these are interpreted as trilinear coordinates, the sum of the negative coordinates exceeds the value of the positive coordinate, and the point retreats to infinity. The same is true for the three cartesian coordinates derived for orthoclase and microcline. Inasmuch as such a repeated treatment is not generally applicable, it is not regarded as usable. The method does appear to be useful, however, for all original positive cartesian coordinates that are interpreted empirically as trilinear or quadriplanar coordinates; and for all such coordinates that conform to the limitations heretofore stated for positive and negative homogeneous coordinates.

References

Carr, G. S. (1886) *A Synopsis of Elementary Results, Pure Mathematics.* Francis Hodgson, London.

Dana, Edward S. (1914) *The System of Mineralogy of James Dwight Dana* (sixth ed.). John Wiley & Sons, Inc., New York, p. 319, 323–325, 407 and 453.

Ferrers, N. M. (1876) *An Elementary Treatise on Trilinear Coordinates, the Method of Reciprocal Polars, and the Theory of Projections* (third ed.). Macmillan and Co., London.

LONEY, S. L. (1923) *The Elements of Coordinate Geometry*, vol. 2, *Trilinear Coordinates.* Macmillan and Co., London.

MERTIE, J. B., JR. (1948) Charting five, six, and seven variables on hypertetrahedral faces. *Am. Mineral.* **33,** 324–336.

——— (1949) Charting five and six variables on the bounding tetrahedra of hypertetrahedra. *Am. Mineral.* **34,** 706–716.

SMITH, CHARLES (1919) *An Elementary Treatise on Conic Sections, by the Methods of Coordinate Geometry.* Macmillan and Co., London, p. 341–377.

SNYDER, VIRGIL AND C. H. SISSAM (1914) *Analytic Geometry of Space.* Henry Holt and Co., New York, p. 109–123.

WASHINGTON, H. S. (1917) Chemical analyses of igneous rocks. *U. S. Geol. Survey Prof. Paper* **99,** 185.

WHITWORTH, W. A. (1866) *Trilinear Coordinates, and Other Methods of Modern Analytical Geometry of Two Dimensions.* Deighton, Bell, and Co., Cambridge, England.

YATES, R. C. (1961) *Analytic Geometry with Calculus.* Prentice-Hall, Inc., Englewood Cliffs, N. J., p. 190–191.

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