A FUN AND EFFECTIVE EXERCISE FOR UNDERSTANDING LATTICES AND SPACE GROUPS

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Introduction

In 1996, Peter Buseck presented a laboratory exercise based on Escher drawings (see MacGillavry 1965) at the Teaching Mineralogy Workshop (Smith College, June 1996). Buseck's exercise (a version of which is included in this book) is an excellent way to learn about two-dimensional symmetry, especially symmetry involving translation. It was fun for all the mineralogists present, but some of its most significant lessons dealt with concepts that may be beyond the scope of a basic mineralogy course. Buseck's ideas excited me, and I recalled that, in 1981, Francois Brisse wrote an article entitled *"La Symétrie bidimensionnelle et le Canada"* (*Two dimensional symmetry and Canada*) which included some beautiful and fascinating color figures based on motifs representative of Canada, the Canadian Provinces and Canadian Territories. Brisse's figures are just as spectacular as Escher's but are simpler for students to analyze.

Brisse's thirteen pictures are brightly colored and contain motifs of fish, boats, flowers, buffalo, polar bears and other things representative of Canada. The lowest symmetry pattern, titled "New Brunswick," has symmetry *p*1 and contains a motif of a Viking ship with sails. A polar bear pattern ("Northwest Territories") has symmetry *p*2 (Figure 1); patterns of maple leafs ("Canada") and pacific dogwood ("British Columbia") belong to tetragonal space groups. In all, Brisse's patterns represent thirteen of the seventeen possible two-dimensional space group symmetries. They exhibit rotational axes, mirror planes, and glide planes. Most unit cells are primitive, but two are centered. The only problem with using Brisse's patterns is that he ignored color when he determined their symmetry. When color is considered, most have less symmetry than Brisse indicates--see the polar bear pattern, below.

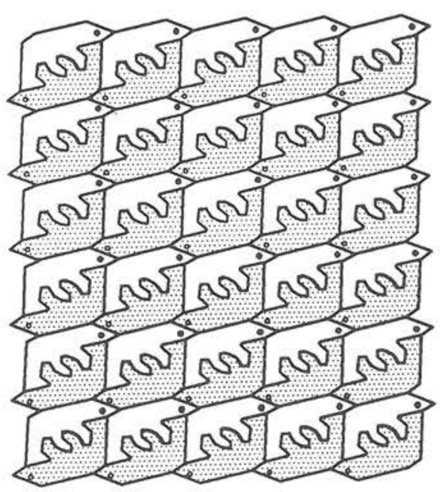

Figure 1. Polar bear pattern modified and redrafted from one of Brisse's figures. Brisse gives the symmetry as *p*2, but if the white and stippled polar bears are considered to be different, it has symmetry *p*1.

The Exercise

The exercise that follows is based on Brisse's figures. My students have found it to be an excellent way to learn about plane symmetry and to understand space groups. Although this exercise only involves two dimensions, students seem to have little difficulty applying the same principles to three dimensions. Before using the exercise, during a semester I would typically give three to five lectures on translational periodicity and space groups. Now, I allow students to spend two class periods on this project and follow with a one hour discussion and wrap-up session. Although some cooperative learning projects seem to be inefficient ways to cover material, this is one example of how group exercises can save time. Students learn the same things more efficiently and have more fun while they are doing it.

Pedagogical Notes

Symmetry can be an intimidating and confusing subject, so students should not be left on their own to do this exercise. They will have the most fun and success if they work in groups of two or three individuals, and if they do all their work in class so they can interact with other groups. I give them minimal instructions possible, but am always present if they have questions. The student groups present their findings and conclusions to the whole class at appropriate times while they are doing the exercise---not just during our wrap-up session. Often I assign different parts of this project to different groups: each group then becomes the experts in something. Different groups sometimes get different results, in part because symmetry is sometimes ambiguous, and consequently we have some lively discussions. In particular, the students like to debate whether Brisse correctly or incorrectly gives the symmetry of his Saskatchewan pattern as *cm*.

This exercise is based on discovery learning. Students need little introduction to lattices and space groups. They can figure things out for themselves. For example, they will figure out what a glide plane is, and if you tell them ahead of time it takes away from the learning experience. The last question, which asks them to make their own symmetrical drawings, is difficult but often leads to some spectacular results.

Canadian Mineralogist originally published the figures for this exercise in color, and the exercise is best done with color reproductions. However, some of Brisse's patterns can be reproduced adequately in black and white. I caution instructors, however, to be aware of applicable copyright laws.

References

Buseck, P.R., 1996, Escher patterns and crystal defects: *Proceedings of the Teaching Mineralogy Workshop*, Smith College, June 1996, p. 44-64.

Brisse, F., 1981, La symétrie bidimensionnelle et le Canada: *The Canadian Mineralogist*, v. 19, p. 217-224.

MacGillavry, C.H., 1965, *Symmetry Aspects of M.C. Escher's Periodic Drawings*: International Union of Crystallography.

Plane Lattices, Space Groups and the Flags of Canada

We have given you *patterns* based on the flags of Canada and Canadian Provinces. They contain *motifs* that systematically repeat to fill two-dimensional space. These drawings come from an article in *Canadian Mineralogist* (vol. 19, pp 217-224, 1981) by François Brisse: *La symétrie bidemensionnelle et le Canada*. The motifs are:

Canada: maple leaf
Prince Edward Island: map of the island
Northwest Territories: polar bear
Nova Scotia: sail boat
British Columbia: pacific dogwood flower
Yukon: fireweed flower
Newfoundland: cod fish
Saskatchewan: wheat sheaf
Ontario: trillium flower
Alberta: wild rose
Manitoba: buffalo
New Brunswick: Viking ship
Quebec: fleur de lis: flower

1. Two-dimensional patterns may have one of seventeen possible symmetries, called two dimensional *point groups*. The Canadian patterns represent thirteen of them. For each:

   a. Put a piece of tracing paper over the pattern. Choose one point on the diagram and put a dot there, and then put dots at all the other identical points on the diagram. (Pay attention to color, the direction something is pointing, etc.--the points must be identical in all ways.) The pattern of points is a *lattice* describing the *translational symmetry* of the pattern.

   b. On your lattice drawing, show all symmetry elements that the lattice has. Use solid lines for mirrors; small lens shapes for 2-fold axes; small triangles for 3-fold axes, small squares for 4-fold axes; small hexagons for 6-fold axes. Also on your drawing, choose two vectors that generate the whole lattice from one initial point. By convention you should choose short vectors, and vectors at special angles like 90°, 60°, etc. if you can.

   c. Choose four near-neighbor lattice points related by the vectors you just chose to define a parallelogram. The four points outline a *unit cell* that repeats many times to make the entire pattern. The vectors you chose, and the lattice, describe the way the unit cells repeat. Draw one unit cell and show all its symmetry using the same symbols as in part b, above.

   d. Finally, put tracing paper over the pattern again, and show all the symmetry elements of the entire pattern. First do this ignoring color (as Brisse did) and then do it again paying attention to color. Neither may yield the same symmetry elements as the lattice. How do the symmetries of the lattice, the unit cell and the entire structure (ignoring color) compare? What if you consider color--how do they compare then?

(Suggestion: you might want to work on all the patterns that appear to have "square" properties first, then go on to the hexagonal or rectangular ones, etc.)

2. Ignoring color, which of the Canadian patterns have 2-fold (or 4-fold or 6-fold, which include 2-fold) axes of symmetry? Which of the lattices have 2-fold (or 4-fold or 6-fold) axes of symmetry? Why are your answers the same, or not the same, for both questions?

3. The two-dimensional patterns can have any of seventeen different symmetries, but unit cells can have only five basic shapes. They correspond to five different lattices with five different symmetries. What are they? What symmetries do they have?

4. According to Brisse, the Canadian patterns all have different symmetries (listed in Table 1; Brisse ignored color when determining symmetry). Consult Table 1 and look at all the patterns with numbers in their symmetry symbols. What do the numbers mean? Similarly, what does *m* mean? And, more difficult, what do the symmetries containing "*g*" have in common? What do you think "*g*" means? Finally, what about the "*p*" or "*c*" that appears as the first element in the symbols--what do they mean? (This is a tough question.) Why do you suppose there is a "1" in some of the symbols?

5. Brisse's drawings are for thirteen of the possible seventeen two-dimensional space groups. Here's a tough task: make your own drawings for the other four space groups. To make your task simpler, and to make it easier to see, use a simple motif composed of circles, dots, squares, etc. Then, for one of the four space groups, try to make something that looks more like one of Brisse's drawings.

Table 1. Plane group symmetries. The table below lists the seventeen possible two-dimensional point groups and space groups. You will find this chart useful for this exercise.

lattice	point group (unit cell symmetry)	space group (structure symmetry)	Canadian patterns
oblique (p)	1	$p1$	New Brunswick
	2	$p2$	Northwest Territories
rectangular (p or c)	m	pm	------
	m	pg	Nova Scotia
	m	cm	Saskatchewan
	$2mm$	$p2mm$	------
	$2mm$	$p2mg$	Manitoba
	$2mm$	$p2gg$	Newfoundland
	$2mm$	$c2mm$	------
square (p)	4	$p4$	Prince Edward Island
	$4mm$	$p4mm$	British Columbia
	$4mm$	$p4gm$	Canada
hexagonal (p)	3	$p3$	------
	$3m$	$p3m1$*	Alberta
	$3m$	$p31m$*	Quebec
	6	$p6$	Ontario
	$6mm$	$p6mm$	Yukon Territories

*In $p3m1$, the *m*irror lines bisect the 60° angles between cell edges; in $p31$m, the reflection lines coincide with the cell edges.